Today's learning goals

- Explain the steps in a proof by mathematical induction
- Use mathematical induction to prove
  - correctness of identities and inequalities
  - properties of algorithms
  - properties of geometric constructions
Induction: a road map

• Today
  • What is a proof by induction?
  • Examples: inequalities, algorithms, constructions

• Thursday
  • Strong induction
  • Recursive definitions: functions, sets, sigma notation
  • More examples / proofs: identities, constructions
Proof strategies so far

**Theorem:** \( \forall x P(x) \) over a given domain.

**Strategy (1):** Let \( x \) be arbitrary element of the domain. \( \text{WTS } P(x) \) is true.

**Strategy (2) if domain finite:** Enumerate all \( x \) in domain. \( \text{WTS } P(x) \) is true.

**Strategy (3) Proof by contradiction:** Assume there is an \( x \) with \( P(x) \) false. \( \text{WTS badness!} \)

**Theorem:** \( \exists x P(x) \) over a given domain.

**Strategy (1):** Define \( x = \ldots \) (some specific element in domain) \( \text{WTS } P(x) \) is true.

**Strategy (2) Proof by contradiction:** Assume that for all \( x \), \( P(x) \) is false. \( \text{WTS badness!} \)

**Theorem:** \( P \rightarrow Q \) over a given domain.

**Strategy (1):** Toward direct proof, assume \( P \) and \( \text{WTS } Q \).

**Strategy (2):** Toward proof by contrapositive, assume \( Q \) is false and \( \text{WTS } P \) is also false.

**Strategy (3) Proof by contradiction:** Assume both \( P \) is true and \( Q \) is false. \( \text{WTS badness!} \)
A new proof strategy

To show that some statement $P(k)$ is true about all nonnegative integers $k$,

1. Show that it’s true about 0 i.e. $P(0)$
2. Show $P(0) \rightarrow P(1)$ Hence conclude $P(1)$
3. Show $P(1) \rightarrow P(2)$ Hence conclude $P(2)$
4. Show $P(2) \rightarrow P(3)$ Hence conclude $P(3)$
5. .....
A new proof strategy

To show that some statement $P(k)$ is true about all nonnegative integers $k$,

1. Show that it’s true about 0, i.e. $P(0)$
2. Show $P(0) \rightarrow P(1)$, hence conclude $P(1)$
3. Show $P(1) \rightarrow P(2)$, hence conclude $P(2)$
4. Show $P(2) \rightarrow P(3)$, hence conclude $P(3)$
5. .....
Mathematical induction

To show that some statement $P(k)$ is true about all nonnegative integers $k$,

1. Show that it’s true about 0 \text{i.e. } P(0)
2. Show $\forall k \ ( P(k) \rightarrow P(k+1) )$ \text{Hence conclude } P(1),…
An inequality

\[(1 + \frac{1}{2})^n \geq 1 + \frac{n}{2}\]

Theorem: This inequality is true for all nonnegative integers.

Proof: by Mathematical Induction.

What's P(k)?

A. \( \left(1 + \frac{1}{2}\right)^k \geq 1 + \frac{k}{2} \)

B. \( \left(1 + \frac{1}{2}\right)^k \geq 1 + \frac{n}{2} \)

C. \( \left(1 + \frac{1}{2}\right)^k \geq 1 + \frac{k}{2} \)

D. \( \left(1 + \frac{n}{2}\right)^k \)

E. None of the above.
An inequality

\[ \left(1 + \frac{1}{2}\right)^n \geq 1 + \frac{n}{2} \]

Theorem: This inequality is true for all nonnegative integers

Proof: by Mathematical Induction.

1. Basis step WTS \( (1 + \frac{1}{2})^0 \geq 1 + \frac{0}{2} \) \( n = 0 \)

2. Inductive step Let \( k \) be a nonnegative integer. Assume \( \text{WTS} \forall k \left( P(k) \rightarrow P(k+1) \right) \)

WTS \( (1 + \frac{1}{2})^{k+1} \geq 1 + \frac{k + 1}{2} \)
Robot

Start at origin, moves on infinite 2-dimensional integer grid. At each step, move to diagonally adjacent grid point.

Can it ever reach (1,0)?
Lemma (Invariant): The sum of the coordinates of any state reachable by the robot is even.

Proof of Lemma: by *mathematical induction* on the number of steps.

Using proof: Sum of coordinates of (1,0) is 1 so not even!
Proof of Invariant

Lemma (Invariant): The sum of the coordinates of any state reachable by the robot is even.

Proof: by Mathematical Induction.

1. **Basis step** WTS The sum of the coordinates reachable after 0 steps is even.

   Start @ origin so coords after 0 steps are (0,0)

   Sum: 0+0=0=2. 0 even

2. **Inductive step** Let k be a nonnegative integer. Assume the sum of the coordinates of any state reachable after k steps is even.

   WTS the sum of the coordinates of any state reachable after k+1 steps is even.
Theorem: If start with an equilateral triangle and divide each side into n equal segments, then connect the division points with all possible line segments parallel to sides of original triangle, then _____ many small triangles will be contained in the original triangle.
Triangles

**Theorem:** At stage $n$, $n^2$ triangles result.

**Proof:** by Mathematical Induction.

1. **Basis step** WTS that when side lengths undivided, $1^2$ triangles are formed.
2. **Inductive step** Let $k$ be a nonnegative integer. Assume that when divide sides into $k$ pieces, $k^2$ triangles result.

   WTS when divide sides into $k+1$ pieces, $(k+1)^2$ triangles result.
Sizes of (finite) sets

If S is a set with exactly \( n \) distinct elements, with \( n \) a nonnegative integer, then S is finite set and \(|S| = n|\).

Which of the following sets are finite?
Assume universe is set of real numbers.

<table>
<thead>
<tr>
<th>Option</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. ( \emptyset )</td>
<td></td>
</tr>
<tr>
<td>B. ( \mathbb{Q} \cap {x \mid 0 \leq x \leq 1} )</td>
<td>( 0, \frac{1}{2}, 0.1 )</td>
</tr>
<tr>
<td>C. ( \mathbb{Z} \cap {x \mid 0 \leq x \leq 1} )</td>
<td>( 0, 1 )</td>
</tr>
<tr>
<td>D. ( \mathbb{Z} \cup {x \mid 0 \leq x \leq 1} )</td>
<td>( 0, 1, -1, -0.1, \frac{1}{2}, \ldots )</td>
</tr>
<tr>
<td>E. None of the above.</td>
<td></td>
</tr>
</tbody>
</table>

An infinite set is a set that is not finite.
Operations on sets

If the sets A, B are finite then

\[
|A \cup B| = ?
\]

A. \(|A| + |B| - |A \cap B|\)
B. \(|A| - |B|\)
C. \(|A| \cdot |B|\)
D. \(|A|^{|B|}\)
E. None of the above.
Operations on sets

If the sets $A$, $B$ are finite then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

If $A$ and $B$ are disjoint ($A \cap B = \emptyset$) then

$$|A \cup B| = |A| + |B|.$$
Operations on sets

If the sets $A$, $B$ are finite then

$$A \times B = \{ (a, b) \mid a \in A \land b \in B \}$$

$$|A \times B| = |A| \cdot |B|$$

$|B|$ many elements for each of the $|A|$ many elements in $A$
How do we prove this general formula?

$$|A \times B| = |A| \cdot |B|$$

How much does $|A \times B|$ change when we add one new element to $A$?

A. Add one element.
B. Add $|B|$ many elements.
C. Multiply number of elements by $|B|$.
D. Raise number of elements to $|B|^{th}$ power.
E. None of the above.
How do we prove this?

**Theorem:** For any nonnegative integer \( n \), if \( A \) is any set of size \( n \), then for any finite set \( B \), \(|A \times B| = |A||B|\).

**Proof by Mathematical Induction:**

1. **Basis step** WTS if \( A \) is empty, then for any finite set \( B \),
   \[
   |A \times B| = |A||B| = 0.
   \]
2. **Inductive step** Let \( k \) be a nonnegative integer. Assume that for any set \( C \) of size \( k \) and any finite set \( D \), \(|C \times D| = |C||D|\).
   
   WTS for any set \( A \) of size \( k+1 \) and any finite set \( B \), \(|A \times B| = |A||B|\).
Induction step

Write $A = \{x_1, x_2, \ldots, x_k, x_{k+1}\} = C \cup \{x_{k+1}\}$

Let $B$ be any finite set. Then

$A \times B = (C \times B) \cup \{(x_{k+1}, y) \mid y \text{ is in } B\}$

so

$|A \times B| = |C \times B| + |B| = k \cdot |B| + |B| = (k+1) \cdot |B| = |A| \cdot |B|$
Operations on sets

Power set: For a set \( S \), its power set is the set of all subsets of \( S \).

\[
P(S) = \{ A \mid A \subseteq S \}
\]

If the set \( S \) is finite then …

Does the size of the power set of \( S \) depend just on the size of \( S \)?
Building up the power set

<table>
<thead>
<tr>
<th>Set</th>
<th>Power set</th>
</tr>
</thead>
<tbody>
<tr>
<td>{a}</td>
<td>{ emptyset, {a} }</td>
</tr>
<tr>
<td>{a,b}</td>
<td>{ emptyset, {a}, {b}, {a,b} }</td>
</tr>
<tr>
<td>{a,b,c}</td>
<td>{ emptyset, {a}, {b}, {a,b}, {c}, {a,c}, {b,c}, {a,b,c} }</td>
</tr>
</tbody>
</table>

**Observe**: the size of the power set increases by a factor of 2 for every new element of the set.

$$|\mathcal{P}(A)| = 2^{|A|}$$
How do we prove this?

**Theorem:** For any nonnegative integer \( n \), if \( A \) is any set of size \( n \), then the power set of \( A \) has size \( 2^n \).

**Proof by Mathematical Induction:**
1. **Basis step** WTS if \( A \) is empty, then its power set has size \( 2^0 \).
2. **Inductive step** Let \( k \) be a nonnegative integer. Assume that the power set of any set \( C \) of size \( k \) has size \( 2^k \).

WTS for any set \( A \) of size \( k+1 \), its power set will have size \( 2^{k+1} \).
Induction step

Write $A = \{x_1, x_2, \ldots, x_k, x_{k+1}\} = C \cup \{x_{k+1}\}$

What are the subsets of $A$?
Reminders

- Discussion section tomorrow
- Review Quiz for this week due tomorrow night
- Office hours may be shifted – check Google calendar
- HW5 due Saturday night

Next class: recursive definitions and variants of induction