Thank you to the TAs + tutors for a speedy grading marathon!
Today's learning goals

• Correctly prove statements using appropriate style conventions, guiding text, notation, and terminology
• Define and differentiate between important sets
• Use correct notation when describing sets: {...}, intervals
• Define and prove properties of: subset relation, power set, Cartesian products of sets, union of sets, intersection of sets, disjoint sets, set differences, complement of a set
Overall strategy

• Do you believe the statement? its negation?
  • Try some small examples.
• Determine logical structure + main connective.
• Determine relevant definitions.
• Map out possible proof strategies.
  • For each strategy: what can we assume, what is the goal?
  • Start with simplest, move to more complicated if/when get stuck.
Some definitions

Set: unordered collection of elements

A = B iff \( \forall x (x \in A \iff x \in B) \)

How to specify these elements?

• Roster \{ ... \}
• Set builder \( \{ x \in U \mid P(x) \} \)

Rosen Sections 2.1, 2.2
Some definitions

A is a subset of B

Subset: \( A \subseteq B \) means \( \forall x(x \in A \rightarrow x \in B) \)

For sets A and B, \( A = B \) if and only if both \( A \subseteq B \) and \( B \subseteq A \)

Why a definition for equality?

\[
\left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}, \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q > 0, \text{gcd}(p, q) = 1 \right\}
\]
Some definitions

Rosen Sections 2.1, 2.2

Subset: \( A \subseteq B \) means \( \forall x (x \in A \rightarrow x \in B) \)

\[ \forall A \forall B [(A = B) \iff (A \subseteq B \wedge B \subseteq A)] \]

Theorem: For sets \( A \) and \( B \), \( A = B \) if and only if both
\[ A \subseteq B \quad \text{and} \quad B \subseteq A \]

Proof: Let \( A, B \) be arbitrary sets. \( \forall \)

1. If \( A = B \), then \( A \subseteq B \) and \( B \subseteq A \)
2. If \( A \subseteq B \) and \( B \subseteq A \), then \( A = B \)

What's the logical structure of this statement?

A. Universal conditional.  
B. Biconditional.  
C. Conjunction (and)  
D. None of the above.
Some definitions

A \not\subseteq B \quad \text{means} \quad \forall x (x \in A \Rightarrow x \not\in B)

Subset: \quad A \subseteq B \quad \text{means} \quad \forall x (x \in A \rightarrow x \in B)

Proper subset means \quad A \mathcal{S} B \quad \text{and} \quad A \not= B

When \ A \ \text{is a subset of} \ B, \ \text{we say} \ B \ \text{is a superset of} \ A.

Some definitions

\begin{align*}
A \not\subseteq B & \quad \text{means} \quad \forall x (x \in A \Rightarrow x \not\in B) \\
\text{Subset:} \quad A \subseteq B & \quad \text{means} \quad \forall x (x \in A \rightarrow x \in B) \\
\text{Proper subset means} \quad A \mathcal{S} B & \quad \text{and} \quad A \not= B \\
\text{When} \ A \ \text{is a subset of} \ B, \ \text{we say} \ B \ \text{is a superset of} \ A.
\end{align*}

Rosen pp. 119-120 Sections 2.1

Which is not true?

A. \( \mathbb{Z}^+ \) is a subset of \( \mathbb{Z} \).
B. \( \mathbb{Z} \) is a subset of \( \mathbb{Q} \).
C. \( \mathbb{Q} \) is a subset of \( \mathbb{R} \).
D. \( \mathbb{R} \) is not a subset of \( \mathbb{Q} \).

Bonus Question:

For any arbitrary \( \mathbb{Z} \), if it's in \( \mathbb{Z}^+ \) then it's a positive so it's an int, so it's in \( \mathbb{Z} \).

Pf: There is an example, \( \sqrt{2} \) of an element of \( \mathbb{R} \) that is not in \( \mathbb{Q} \).
Some definitions

Empty set: $\emptyset = \{\} = \{x \mid x \neq x\}$

Which of the following is **not** true?

A. $\emptyset \subseteq \emptyset$ is true
B. $\emptyset \in \emptyset$ is **not** true bec $\emptyset$ has no elements.
C. For any set $A$, $\emptyset \subseteq A$ is true
D. For some set $B$, $\emptyset \in B$
E. More than one of the above.

For all sets $B$, $\emptyset \in B$ if by counterexample $\emptyset$ or $\{1\}$.
Operations on sets

Power set: For a set $S$, its power set is the set of all subsets of $S$. 

$$P(S) = \{ A \mid A \subseteq S \}$$

$P(\{2\})$

$$\{2\} \subseteq \{2\}$$

$$\emptyset = \{ \} \subseteq \{2\}$$

$$P(\{2\}) = \{ \emptyset, \{2\} \}$$
Operations on sets

Power set: For a set $S$, its power set is the set of all subsets of $S$.

$\mathcal{P}(S) = \{A \mid A \subseteq S\}$

Which of the following is not necessarily (always) true?

A. $S \in \mathcal{P}(S)$
B. $\emptyset \in \mathcal{P}(S)$
C. $\emptyset \subseteq \mathcal{P}(S)$
D. $\emptyset \in S$
E. None of the above.

- A. True
- B. True
- C. True
- D. False
- E. False
Operations on sets

Given two sets $A$, $B$ we can define

- $A \cap B$: Intersection of $A$ and $B$
- $A \cup B$: Union of $A$ and $B$
- $A - B$: Difference of $A$ and $B$
- $A \times B$: Cartesian product of $A$ and $B$
Operations on sets

Given two sets $A$, $B$ we can define

$A \cap B = \{ x \mid x \in A \land x \in B \}$

$A \cup B = \{ x \mid x \in A \lor x \in B \}$

$A - B = \{ x \mid x \in A \land x \notin B \}$

$A \times B = \{(x, y) \mid x \in A \land y \in B \}$

Examples
Operations on sets

Given two sets A, B we can define

\[ A \cap B = \{ x \mid x \in A \land x \in B \} \]
\[ A \cup B = \{ x \mid x \in A \lor x \in B \} \]
\[ A - B = \{ x \mid x \in A \land x \notin B \} \]
\[ A \times B = \{ (x, y) \mid x \in A \land y \in B \} \]

Which of these is true?

A. \( A \cap B = B \cap A \)
B. \( A \cup B = B \cup A \)
C. \( A - B = B - A \)
D. \( A \times B = B \times A \)
E. None of the above.
Operations on sets

Given two sets A, B we can define

\[ A \cap B = \{ x \mid x \in A \land x \in B \} \]
\[ A \cup B = \{ x \mid x \in A \lor x \in B \} \]
\[ A - B = \{ x \mid x \in A \land x \notin B \} \]
\[ A \times B = \{ (x, y) \mid x \in A \land y \in B \} \]

Which of the following can't be labelled in this Venn diagram?

A. \( A \cap B \)
B. \( A \cup B \)
C. \( A - B \)
D. \( A \times B \)
E. None of the above.
Operations on sets

Given two sets \( A, B \) we can define

\[
A \cap B = \{ x \mid x \in A \land x \in B \}
\]

\[
A \cup B = \{ x \mid x \in A \lor x \in B \}
\]

\[
A - B = \{ x \mid x \in A \land x \notin B \}
\]

When can you conclude that \( A \) is a subset of \( B \)?

A. When \( A \cup B = A \)
B. When \( A - B = A \)
C. When \( A - B = B - A \)
D. When \( A \cap B = A \)
E. None of the above.
Some definitions

**Disjoint sets**: two sets are disjoint if their intersection is the empty set.

\[
\{1,2,3\} \cap \{4,5,6\} = \emptyset
\]

If \( A \cap B = \emptyset \) then \( A - B = A \) and \( B - A = B \)
Disjoint sets: two sets are disjoint if their intersection is the empty set.

Which of the following is not true?
A. $\emptyset \subseteq \emptyset$
B. $\emptyset \in \emptyset$
C. For any set $A$, $\emptyset \subseteq A$
D. For some set $B$, $\emptyset \in B$
E. More than one of the above.
Generalized union / intersection  

Union and intersection are associative

\[ A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i \]

\[ A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i \]
Analogue: Summation notation

The expression is:

\[ a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j \]

- **Index of summation**
- **Lower limit**
- **Upper limit**

Ex (1): Write an expression for the sum of the first \( n \) positive integers.
Ex (2): Write an expression for the sum of the first \( n \) positive even integers.
Ex (3): Write an expression for the sum of the first \( n \) positive odd integers.