CSE 20
DISCRETE MATH

Fall 2017

http://cseweb.ucsd.edu/classes/fa17/cse20-ab/
Today's learning goals

- Correctly prove statements using appropriate style conventions, guiding text, notation, and terminology
- Define and differentiate between important sets
- Use correct notation when describing sets: {...}, intervals
- Define and prove properties of: subset relation, power set, Cartesian products of sets, union of sets, intersection of sets, disjoint sets, set differences, complement of a set
Overall strategy

• Do you believe the statement? its negation?
  • Try some small examples.
• Determine logical structure + main connective.
• Determine relevant definitions.
• Map out possible proof strategies.
  • For each strategy: what can we **assume**, what is the **goal**?
  • Start with simplest, move to more complicated if/when get stuck.

Direct proof, construction, exhaustive, etc.
Contradiction, hidden cases
Some definitions

Set: unordered collection of elements

\( A = B \iff \forall x (x \in A \iff x \in B) \)

How to specify these elements?

- Roster \( \{ \ldots \} \)
- Set builder \( \{ x \in U \mid P(x) \} \)
Some definitions

\[ A \subseteq B \]

means \[ \forall x (x \in A \rightarrow x \in B) \]

For sets A and B, A = B if and only if both
\[ A \subseteq B \quad \text{and} \quad B \subseteq A \]

Why a definition for equality?

\[ \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\} \quad \text{and} \quad \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q > 0, \gcd(p, q) = 1 \right\} \]
Some definitions

Subset: \( A \subseteq B \) means \( \forall x (x \in A \rightarrow x \in B) \)

Theorem: For sets \( A \) and \( B \), \( A = B \) if and only if both
\( A \subseteq B \) and \( B \subseteq A \)

Proof:

Let \( A \), \( B \) arbitrary sets. \( \text{WTS} \)

1. \( A = B \rightarrow [\forall x (x \in A \rightarrow x \in B) \land \forall x (x \in B \rightarrow x \in A)] \)
2. \( (A \subseteq B \land B \subseteq A) \rightarrow A = B \)

What's the logical structure of this statement?
A. Universal conditional.  
B. Biconditional.  
C. Conjunction (and)  
D. None of the above.
Some definitions

Subset: \( A \subseteq B \) means \( \forall x (x \in A \Rightarrow x \in B) \)

Proper subset means \( A \subseteq B \) and \( A \neq B \)

When A is a subset of B, we say B is a superset of A.

Which is not true?

A. \( \mathbb{Z}^+ \) is a subset of \( \mathbb{Z} \).
B. \( \mathbb{Z} \) is a subset of \( \mathbb{Q} \).
C. \( \mathbb{Q} \) is a subset of \( \mathbb{R} \).
D. \( \mathbb{R} \) is not a subset of \( \mathbb{Q} \).

Pf: Let \( x \) be arbitrary. Towards direct

assume \( x \in \mathbb{Z}^+ \). By def of \( \mathbb{Z}^+ \), \( x \in \mathbb{Z} \).
Empty set: \( \emptyset = \{ \} = \{ x \mid x \neq x \} \)

Which of the following is **not** true?

A. \( \emptyset \subseteq \emptyset \)
   - **T** \( \text{Pf: } \emptyset = \emptyset \text{ so apply then that says equality is not a subset.} \)

B. \( \emptyset \in \emptyset \)
   - **F** \( \text{WTS } \forall x (x \notin \emptyset) \)

C. For any set A, \( \emptyset \subseteq A \)
   - **T** \( \text{Pf: WTS } \forall x (x \in \emptyset \implies x \in A) \)

D. For some set B, \( \emptyset \in B \)
   - **T** \( \text{Pf: By ex eg. } B = \{ \emptyset \} \)

E. More than one of the above.
Operations on sets

Power set: For a set S, its power set is the set of all subsets of S.

$$\mathcal{P}(S) = \{ A \mid A \subseteq S \}$$

$$\mathcal{P}\{2\} = \{ \emptyset, \{2\} \}$$

$$\{2, \emptyset\} \subseteq \{2\} ? \quad \text{No} \quad \emptyset \not\subseteq \{2\}.$$
Operations on sets

Rosen Sections 2.1, 2.2

Power set: For a set \( S \), its power set is the set of all subsets of \( S \).

\[ \mathcal{P}(S) = \{ A \mid A \subseteq S \} \]

Which of the following is not necessarily (always) true?
A. \( S \in \mathcal{P}(S) \)
   - Pf: WTS \( S \subseteq S \)
   - \( \checkmark \)
B. \( \emptyset \in \mathcal{P}(S) \)
   - Pf: WTS \( \emptyset \subseteq S \)
   - \( \checkmark \)
C. \( \emptyset \subseteq \mathcal{P}(S) \)
   - Pf: \( \mathcal{P}(S) \) is a set. Already shown \( \forall A (\emptyset \subseteq A) \)
   - \( \checkmark \)
D. \( \emptyset \in S \)
   - counterexamples: \( \emptyset, \{2\}, \mathbb{Z} \)
   - \( \checkmark \)
E. None of the above.
Operations on sets

Given two sets A, B we can define

\[ A \cap B \]  Intersection of A and B

\[ A \cup B \]  Union of A and B

\[ A - B \]  Difference of A and B

\[ A \times B \]  Cartesian product of A and B
Operations on sets

Given two sets $A$, $B$ we can define

$$A \cap B = \{x \mid x \in A \land x \in B\}$$

$$A \cup B = \{x \mid x \in A \lor x \in B\}$$

$$A - B = \{x \mid x \in A \land x \notin B\}$$

$$A \times B = \{(x, y) \mid x \in A \land y \in B\}$$

Examples
Operations on sets

Given two sets $A$, $B$ we can define

$$A \cap B = \{ x \mid x \in A \land x \in B \}$$
$$A \cup B = \{ x \mid x \in A \lor x \in B \}$$
$$A - B = \{ x \mid x \in A \land x \notin B \}$$
$$A \times B = \{ (x, y) \mid x \in A \land y \in B \}$$

Which of these is true?

A. $A \cap B = B \cap A$
B. $A \cup B = B \cup A$
C. $A - B = B - A$
D. $A \times B = B \times A$
E. None of the above.
Operations on sets

Rosen Sections 2.1, 2.2

Given two sets $A$, $B$ we can define

\[ A \cap B = \{x \mid x \in A \land x \in B\} \]
\[ A \cup B = \{x \mid x \in A \lor x \in B\} \]
\[ A - B = \{x \mid x \in A \land x \notin B\} \]
\[ A \times B = \{(x, y) \mid x \in A \land y \in B\} \]

Which of the following can't be labelled in this Venn diagram?

A. $A \cap B$
B. $A \cup B$
C. $A - B$
D. $A \times B$
E. None of the above.
Operations on sets

Given two sets $A, B$ we can define

$A \cap B = \{x \mid x \in A \land x \in B\}$
$A \cup B = \{x \mid x \in A \lor x \in B\}$
$A - B = \{x \mid x \in A \land x \notin B\}$

When can you conclude that $A$ is a **subset** of $B$?

A. When $A \cup B = A$
B. When $A - B = A$
C. When $A - B = B - A$
D. When $A \cap B = A$
E. None of the above.
Some definitions

**Disjoint sets**: two sets are disjoint if their intersection is the empty set.

*Examples?*
Some definitions

Disjoint sets: two sets are disjoint if their intersection is the empty set

Which of the following is not true?
A. $\emptyset \subseteq \emptyset$
B. $\emptyset \in \emptyset$
C. For any set A, $\emptyset \subseteq A$
D. For some set B, $\emptyset \in B$
E. More than one of the above.
Generalized union / intersection

Union and intersection are associative

\[ A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i \]

\[ A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i \]

Rosen p. 132
Analogue: Summation notation

\[ a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j \]

*Index of summation*
*Lower limit*
*Upper limit*

Ex (1): Write an expression for the sum of the first \( n \) positive integers.
Ex (2): Write an expression for the sum of the first \( n \) positive even integers.
Ex (3): Write an expression for the sum of the first \( n \) positive odd integers.