Discussion 8

1. Is each of the following sets finite or infinite?
   (a) The negative integers
   (b) The even integers
   (c) The integers less than 100
   (d) The real numbers between 0 and \( \frac{1}{2} \)
   (e) The positive integers less than 1,000,000
   (f) The integers that are multiples of 7
   (g) The integers that are divisors of 2017
   (h) The set \( \{2, 3\} \times \mathbb{Z}^+ \)
   (i) The set \( \{2, 3\} \cap \mathbb{Z}^+ \)
   (j) The set \( \{2, 3\} \cup \mathbb{Z}^+ \)

2. Show that for any (nonempty) sets \( A, B, C \) and any functions \( f : A \to B \) and \( g : B \to C \), if \( f \) and \( g \) are one-to-one, then \( g \circ f \) is also one-to-one.
   Use this lemma to prove that if \( |A| \leq |B| \) and \( |B| \leq |C| \) then \( |A| \leq |C| \).
3. In Cantor’s diagonalization argument, we show that any function from a set to its power set is not a bijection. To get a better feel for this argument, show why each of the following specific functions are not bijections. Is each function one-to-one? Is each function onto?

(a) \( f : \{a, b, c\} \to \mathcal{P}(\{a, b, c\}) \) given by

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
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<tbody>
<tr>
<td>( a )</td>
<td>( \emptyset )</td>
</tr>
<tr>
<td>( b )</td>
<td>( {a, b, c} )</td>
</tr>
<tr>
<td>( c )</td>
<td>( {a, b} )</td>
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(b) \( f : \mathbb{N} \to \mathcal{P}(\mathbb{N}) \) given by \( f(n) = \{0, 1, \ldots, n\} \).

(c) \( f : \{0, 1\}^* \to \mathcal{P}(\{0, 1\}^*) \) given recursively by \( f(\lambda) = \emptyset \) and \( f(wx) = f(w) \cup \{x\} \).