1.1 (5 points) In this problem, we will use the construction in the first part of the proof of Theorem 1.39 in Sipser which shows that every NFA (without \( \varepsilon \) transitions) has an equivalent DFA. Use the construction to create an equivalent DFA \( D \) from the following NFA \( N \) and provide the state diagram of the result. You may leave out unreachable states.

(An automaton \( A \) is equivalent to \( B \) if \( L(A) = L(B) \))
Solution 1.1
1.2 (5 points) Describe \( L(N) \) and provide the state diagram of a DFA \( D' \) equivalent to \( N \) not necessarily using the construction.

Solution 1.2 \( L(N) = 0^*(0 \cup 1)1^*1 \)
2. **(10 points)** In this problem we will prove that every Multi-Start-State-NFA (MSSNFA) has an equivalent NFA with a single start state.
For this problem we require you to:

1. Provide the construction part of the proof from an MSSNFA $M$ to an NFA $N$ using the formal definition of an MSSNFA and the formal definition of an NFA.

2. Perform the construction on the following MSSNFA $M$ and provide the state diagram of the result.

3. Use the formal definition of computation to show that $L(M) = L(N)$.
   This requires two parts: $w \in L(M) \rightarrow w \in L(N)$ and $w \in L(N) \rightarrow w \in L(M)$.

See Sipser for the formal definition of an NFA.

The following is the definition of a Multi-Start-State-NFA and its definition of computation.

A Multi-Start-State-NFA is a 5-tuple $(Q, \Sigma, \delta, Q_0, F)$ where:

1. $Q$ is a finite set of states
2. $\Sigma$ is a finite alphabet
3. $\delta : Q \times \Sigma \rightarrow \mathcal{P}(Q)$ is a transition function
4. $Q_0 \subseteq Q$ is the set of start states
5. $F \subseteq Q$ is the set of final states

The formal definition of computation for a Multi-Start-State-NFA is similar to that of an NFA. Let $M = (Q, \Sigma, \delta, Q_0, F)$ be a Multi-Start-State-NFA and $w$ a string over the alphabet $\Sigma$. Then we say that $M$ accepts $w$ if we can write $w$ as $w = y_1 y_2 \ldots y_m$, where $y_i$ is a member of $\Sigma$ and a sequence of states $r_0, r_1, \ldots, r_m$ exists in $Q$ with three conditions:

1. $r_0 \in Q_0$
2. $r_{i+1} \in \delta(r_i, y_{i+1})$, for $i = 0, \ldots, m - 1$
3. $r_m \in F$. 


Solution 2  Given MSSNFA $M = (Q, \Sigma, \delta, Q_0, F)$ construct NFA $N = (Q', \Sigma, \delta', q', F)$ where:

- $q' \not\in Q$
- $Q' = Q \cup \{q'\}$
- $\delta'(q', c) = \begin{cases} 
Q_0 & q = q' \text{ and } c = \varepsilon \\
\emptyset & q = q' \text{ and } c \neq \varepsilon \\
\delta(q, c) & \text{otherwise}
\end{cases}$
Solution 2(cont)
WTS: \( w \in L(M) \rightarrow w \in L(N) \)
Let \( w \in L(M) \).
There must be a way to write \( w \) as \( w = y_2 y_3 \ldots y_m \) where \( y_i \in \Sigma \) and there is a sequence of states \( r_1 r_2 \ldots r_m \in Q \) where:

1. \( r_1 \in Q_0 \)
2. \( r_{i+1} \in \delta(r_i, y_{i+1}) \), for \( i = 1 \ldots m - 1 \)
3. \( r_m \in F \)

We can also write \( w \) as \( w = y_1 y_2 y_3 \ldots y_m \) where \( y_1 = \varepsilon \).
Consider the sequence of states \( r_0 r_1 r_2 \ldots r_m \in Q \cup \{q'\} \) where \( r_0 = q' \), \( r_1 \in Q_0 \) and \( Q_0 = \delta(q', \varepsilon) \) so \( r_1 \in \delta(r_0, y_1) \).
The following conditions hold:

1. \( r_0 = q' \)
2. \( r_{i+1} \in \delta(r_i, y_{i+1}) \), for \( i = 0 \ldots m - 1 \)
3. \( r_m \in F \)

So \( N \) accepts \( w \).

WTS: \( w \in L(N) \rightarrow w \in L(M) \)
Let \( w \in L(N) \).
There must be a way to write \( w \) as \( w = y_1 y_2 y_3 \ldots y_m \) where \( y_i \in \Sigma \) and there is a sequence of states \( r_0 r_1 \ldots r_m \in Q \cup \{q'\} \) where:

1. \( r_0 = q' \)
2. \( r_{i+1} \in \delta(r_i, y_{i+1}) \), for \( i = 0 \ldots m - 1 \)
3. \( r_m \in F \)

The only non-empty outgoing transitions from \( q' \) are epsilon transitions so \( r_0 = \varepsilon \).
Since there are no transitions to \( q', q_1, \ldots, r_m \in Q \).
Since \( \delta(q', \varepsilon) = Q_0 \), \( r_1 \in Q_0 \).
We can also write \( w \) as \( w = y_2 y_3 \ldots y_m \) and there is a sequence of states \( r_1 r_2 \ldots r_m \in Q \) where:

1. \( r_1 \in Q_0 \)
2. \( r_{i+1} \in \delta(r_i, y_{i+1}) \), for \( i = 1 \ldots m - 1 \)
3. \( r_m \in F \)

So \( M \) accepts \( w \).
3. **(10 points)** Prove that regular languages are closed under the *reverse* operation. Use the following definitions:

- The **reverse** of $w$, written $w^R$, is the string obtained by writing $w$ in the opposite order (i.e., $w_n w_{n-1} \ldots w_1$)
- $\text{reverse}(L) = \{w^R \mid w \in L\}$

For this problem we require you to:

1. Provide the construction part of the proof from a NFA $N$ to an NFA $N'$ using formal definitions.
2. Perform the construction on the following NFA $N$ and provide the state diagram of the result.

(You may construct a MSSNFA $M$ instead if you want.)
(The proof of correctness is not required.)
Solution 3  Given NFA $N = (Q, \Sigma, \delta, q_0, F)$ construct MSSNFA $M = (Q, \Sigma, \delta', Q_0, F')$ where

- $\delta'(q, c) = \{ q' \mid q \in \delta(q', c) \}$
- $Q_0 = F$
- $F' = \{ q_0 \}$
4. (10 points) Prove that regular languages are closed under the stutter operation.
Use the following definition: \( \text{stutter}(L) = \{a_1a_1a_2a_2...a_na_n \mid w = a_1a_2...a_n \in L \text{ and } a_i \in \Sigma \} \)
For this problem we require you to:

1. Provide the construction part of the proof from a DFA \( D \) to an NFA \( N \) using formal definitions.

2. Perform the construction on the following state diagram and provide the state diagram of the result.

You may construct a DFA \( D' \) instead if you want.
(The proof of correctness is not required.)

Solution 4  Given DFA \( D = (Q, \Sigma, \delta, q_0, F) \) construct NFA \( N = (Q', \Sigma, \delta', q'_0, F') \) where

- \( Q' = Q \times \Sigma_\epsilon \)
- \( \delta'((q,c),a) = \begin{cases} 
(q, \epsilon) & a = c \neq \epsilon \\
\delta(q,a) & c = \epsilon \neq a \\
\emptyset & \text{otherwise} 
\end{cases} \)
- \( q'_0 = (q_0, \epsilon) \)
- \( F' = F \times \{\epsilon\} \)
5. (10 points) Prove that regular languages are closed under the *skip* operation.
Use the following definition: \( \text{skip}(L) = \{ a_1a_3a_5...a_n \mid w = a_1a_2...a_n \in L \text{ and } n \text{ is odd and } a_i \in \Sigma \} \)
For this problem we require you to:

1. Provide the construction part of the proof from a DFA \( D \) to an NFA \( N \) using formal definitions.
2. Perform the construction on the following DFA \( D \) and provide the state diagram of the result.

(The proof of correctness is not required.)
Solution 5  Given DFA $D = (Q, \Sigma, \delta, q_0, F)$ construct NFA $N = (Q', \Sigma, \delta', q', F)$ where

- $q' \notin Q'$
- $Q' = Q \cup \{q'\}$
- $\delta'(q, a) = \begin{cases} 
\{\delta(q_0, a)\} & a \neq \varepsilon \text{ and } q = q' \\
\{\delta(\delta(q, c), a) \mid c \in \Sigma\} & a \neq \varepsilon \text{ and } q \in Q \\
\emptyset & \text{otherwise} 
\end{cases}$