When sum rule fails

Let $A = \{\text{people who know Java}\}$ and $B = \{\text{people who know C}\}$

$\# \text{people who know Java or C} = \# \text{people who know Java}$
When sum rule fails

Let $A = \{ \text{people who know Java} \}$ and $B = \{ \text{people who know C} \}$

$\# \text{people who know Java or C} = \# \text{people who know Java} + \# \text{people who know C}$

Double counted!
Let $A = \{ \text{people who know Java} \}$ and $B = \{ \text{people who know C} \}$

$\# \text{people who know Java or C} = \# \text{people who know Java} + \# \text{people who know C} - \# \text{people who know both}$
Inclusion-Exclusion principle

Let $A = \{ \text{people who know Java} \}$ and $B = \{ \text{people who know C} \}$

$$|A \cup B| = |A| + |B| - |A \cap B|$$
Inclusion-Exclusion for three sets

\[ |A \cup B \cup C| =? \]

Rosen p. 392-394
Inclusion-Exclusion for three sets

|A ∪ B ∪ C| =?
Inclusion-Exclusion for three sets

\[ |A \cup B \cup C| = ? \]
Inclusion-Exclusion for three sets

\[ |A \cup B \cup C| = ? \]
Inclusion-Exclusion for three sets

\[ |A \cup B \cup C| = ? \]
Inclusion-Exclusion for three sets

\[ |A \cup B \cup C| = ? \]
Inclusion-Exclusion for three sets

\[ |A \cup B \cup C| =? \]
Inclusion-Exclusion for three sets

\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C| \]
Inclusion-Exclusion principle

If $A_1, A_2, \ldots, A_n$ are finite sets then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k|$$

$$- \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$
If \( A = X_1 \cup X_2 \cup \ldots \cup X_n \) and all \( X_i, X_j \) disjoint and all \( X_i \) have same size, then

\[ |X_i| = \frac{|A|}{n} \]

More generally:

There are \( \frac{n}{d} \) ways to do a task if it can be done using a procedure that can be carried out in \( n \) ways, and for every way \( w \), \( d \) of the \( n \) ways give the same result as \( w \) did.
If $A = X_1 \cup X_2 \cup \ldots \cup X_n$ and all $X_i, X_j$ disjoint and all $X_i$ have same size, then

$$|X_i| = |A| / n$$

More generally:

There are $\frac{n}{d}$ ways to do a task if it can be done using a procedure that can be carried out in $n$ ways, and for every way $w$, $d$ of the $n$ ways give the same result as $w$ did.
If \( A = X_1 \cup X_2 \cup \ldots \cup X_n \) and all \( X_i, X_j \) disjoint and all \( X_i \) have same size, then

\[
|X_i| = |A| / n
\]

Or in other words,

If objects are partitioned into categories of equal size, and we want to think of different objects as being the same if they are in the same category, then

\[
\text{# categories} = (\text{# objects}) / (\text{size of each category})
\]
An ice cream parlor has n different flavors available. How many ways are there to order a two-scoop ice cream cone (where you specify which scoop goes on bottom and which on top, and the two flavors must be different)?

A. $n^2$
B. $n!$
C. $n(n-1)$
D. $2n$
E. None of the above.
An ice cream parlor has $n$ different flavors available. How can we use our earlier answer to decide the number of cones, if we count two cones as the same if they have the same two flavors (even if they're in opposite order)?

A. Double the previous answer.
B. Divide the previous answer by 2.
C. Square the previous answer.
D. Keep the previous answer.
E. None of the above.
Ice cream!

An ice cream parlor has $n$ different flavors available. How can we use our earlier answer to decide the number of cones, if we count two cones as the same if they have the same two flavors (even if they're in opposite order)?

**Objects:**
**Categories:**
**Size of each category:**

$$\text{# categories} = \frac{\text{# objects}}{\text{(size of each category)}}$$
Ice cream!

An ice cream parlor has \( n \) different flavors available. How can we use our earlier answer to decide the number of cones, if we count two cones as the same if they have the same two flavors (even if they're in opposite order)?

**Objects**: cones

**Categories**: flavor pairs (regardless of order)

**Size of each category**:

\[
\text{# categories} = \frac{\text{# objects}}{\text{size of each category}}
\]
An ice cream parlor has \( n \) different flavors available. How can we use our earlier answer to decide the number of cones, if we count two cones as the same if they have the same two flavors (even if they're in opposite order)?

**Objects:** cones \( n(n-1) \)

**Categories:** flavor pairs (regardless of order)

**Size of each category:** 2

\[
\text{# categories} = \frac{(n)(n-1)}{2}
\]

Avoiding double-counting
How many different colored triangles can we create by tying these three pipe cleaners end-to-end?

A. $3!$
B. $2^3$
C. $3^2$
D. 1
E. None of the above.
How many different colored triangles can we create by tying these three pipe cleaners end-to-end?

**Objects**: all different colored triangles

**Categories**: physical colored triangles (two triangles are the same if they can be rotated and/or flipped to look alike)

**Size of each category**:

\[
\# \text{ categories} = \frac{\# \text{ objects}}{\text{size of each category}}
\]
How many different colored triangles can we create by tying these three pipe cleaners end-to-end?

**Objects**: all different colored triangles \(3!\)

**Categories**: physical colored triangles (two triangles are the same if they can be rotated and/or flipped to look alike)

**Size of each category**: \((3)(2)\) three possible rotations, two possible flips

\[
\text{# categories} = \frac{\text{(# objects)}}{\text{(size of each category)}} = \frac{6}{6} = 1
\]
Object Symmetries
How many length $n$ binary strings contain $k$ ones?

For example, $n=6$ $k=4$

Which of these strings matches this example?

A. 101101
B. 1100011101
C. 111011
D. 1101
E. None of the above.

Density is number of ones

Rosen p. 413
How many length $n$ binary strings contain $k$ ones?

For example, $n=6$ $k=4$

Product rule: How many options for the first bit? the second? the third?
How many length $n$ binary strings contain $k$ ones?

For example, $n=6$ $k=4$

Tree diagram: *gets very big & is hard to generalize*
Fixed-density Binary Strings

How many length \( n \) binary strings contain \( k \) ones?

For example, \( n=6 \) \( k=4 \)

Another approach: use a different representation i.e. count with categories

Objects:
Categories:
Size of each category:

\[
# \text{ categories} = \frac{# \text{ objects}}{(\text{size of each category})}
\]

Rosen p. 413
How many length \( n \) binary strings contain \( k \) ones?

For example, \( n=6 \) \( k=4 \)

Another approach: **use a different representation** i.e. count with categories

**Objects:** all strings made up of \( 0_1, 0_2, 1_1, 1_2, 1_3, 1_4 \)
**Categories:** strings that agree except subscripts
**Size of each category:** \( \text{Subscripts so objects are distinct} \)

\[ \# \text{ categories} = \frac{\# \text{ objects}}{\text{size of each category}} \]
How many length \( n \) binary strings contain \( k \) ones?

For example, \( n=6 \) \( k=4 \)

Another approach: use a different representation i.e. count with categories

**Objects**: all strings made up of \( 0_1, 0_2, 1_1, 1_2, 1_3, 1_4 \) \( 6! \)

**Categories**: strings that agree except subscripts

**Size of each category**: ?

\[
\# \text{ categories} = \frac{\# \text{ objects}}{\text{size of each category}}
\]
How many subscripted strings i.e. rearrangements of the symbols

\[ 0_1, 0_2, 1_1, 1_2, 1_3, 1_4 \]

result in

\[ 101101 \]

when the subscripts are removed?

A. 6!
B. 4!
C. 2!
D. 4!2!
E. None of the above.
How many length $n$ binary strings contain $k$ ones?

For example, $n=6$ $k=4$

Another approach: **use a different representation** i.e. count with categories

**Objects**: all strings made up of $0_1$, $0_2$, $1_1$, $1_2$, $1_3$, $1_4$  
**Categories**: strings that agree except subscripts  
**Size of each category**: $4!2!$

$\#$ categories $= (\#$ objects) / (size of each category)  
$= 6! / (4!2!)$
How many length $n$ binary strings contain $k$ ones?

Another approach: use a different representation i.e. count with categories

**Objects**: all strings made up of $0_1, 0_2, \ldots, 0_{n-k}, 1_1, 1_2, \ldots, 1_k$  
\[ n! \]

**Categories**: strings that agree except subscripts  
**Size of each category**:  
\[ k! (n-k)! \]

\[
\text{# categories} = \frac{\text{# objects}}{\text{size of each category}} \\
= \frac{n!}{k! (n-k)!}
\]
A permutation of $r$ elements from a set of $n$ distinct objects is an ordered arrangement of them. There are

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1)$$

many of these.

A combination of $r$ elements from a set of $n$ distinct objects is an unordered selection of them. There are

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

many of these.
How many length $n$ binary strings contain $k$ ones?

How to express this using the new terminology?

A. $C(n,k)$
B. $C(n,n-k)$
C. $P(n,k)$
D. $P(n,n-k)$
E. None of the above
How many length \( n \) binary strings contain \( k \) ones?

How to express this using the new terminology?

A. \( C(n,k) \)  
{1,2,3..n} is set of positions in string, choose \( k \) positions for 1s  

B. \( C(n,n-k) \)  
{1,2,3..n} is set of positions in string, choose \( n-k \) positions for 0s

C. \( P(n,k) \)  

D. \( P(n,n-k) \)  

E. None of the above
An ice cream parlor has n different flavors available.

How many ice cream cones are there, if we count two cones as the same if they have the same two flavors (even if they're in opposite order)?

**Objects:** cones \( n(n-1) \)

**Categories:** flavor pairs (regardless of order)

**Size of each category:** 2

\[
\# \text{ categories} = \frac{n(n-1)}{2}
\]

Order doesn't matter so selecting a subset of size 2 of the n possible flavors:

\[
C(n,2) = \frac{n!}{2! (n-2)!} = \frac{n(n-1)}{2}
\]
Binomial: sum of two terms, say \(x\) and \(y\).

What do powers of binomials look like?

\[
(x+y)^4 = (x+y)(x+y)(x+y)(x+y) \\
= (x^2+2xy+y^2)(x^2+2xy+y^2) \\
= x^4+4x^3y+6x^2y^2+4xy^3+y^4
\]

In general, for \((x+y)^n\)

A. All terms in the expansion are \((\text{some coefficient times}) x^k y^{n-k}\) for some \(k\), \(0\leq k\leq n\).

B. All coefficients in the expansion are integers between 1 and \(n\).

C. There is symmetry in the coefficients in the expansion.

D. The coefficients of \(x^n\) and \(y^n\) are both 1.

E. All of the above.
Binomial Theorem

Rosen p. 416

\[(x+y)^n = (x+y)(x+y)\ldots(x+y)\]

\[= x^n + \_\_x^{n-1}y + \_\_x^{n-2}y^2 + \ldots + \_\_x^ky^{n-k} + \ldots + \_\_x^2y^{n-2} + \_\_xy^{n-1} + y^n\]

Number of ways we can choose \(k\) of the \(n\) factors (to contribute to \(x\)) and hence also \(n-k\) of the factors (to contribute to \(y\))
Binomial Theorem

\[(x+y)^n = (x+y)(x+y)\ldots(x+y)\]

\[= x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \ldots + \binom{n}{k} x^k y^{n-k} + \ldots + \binom{n}{k-1} xy^{n-2} + xy^{n-1} + y^n\]

Number of ways we can choose \(k\) of the \(n\) factors (to contribute to \(x\))
and hence also \(n-k\) of the factors (to contribute to \(y\)) \(\binom{n}{k}\)

\[= x^n + \binom{n}{1} x^{n-1}y + \ldots + \binom{n}{k} x^k y^{n-k} + \ldots + \binom{n}{k-1} xy^{n-2} + xy^{n-1} + y^n\]
What's an identity? An equation that is always true.

To prove $LHS = RHS$

- Use algebraic manipulations of formulas.

OR

- Interpret each side as counting some collection of strings, and then prove a statement about those sets of strings.
Theorem: \[ \binom{n}{k} = \binom{n}{n-k} \] Rosen p. 411
Theorem: \[ \binom{n}{k} = \binom{n}{n-k} \]

Proof 1: Use formula
\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}
\]
Symmetry Identity

Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof 1: Use formula

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}
\]

Proof 2: Combinatorial interpretation?

LHS counts number of binary strings of length n with k ones
RHS counts number of binary strings of length n with n-k ones
Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof 1: Use formula
\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}
\]

Proof 2: Combinatorial interpretation?

**LHS** counts number of binary strings of length n with k ones and n-k zeros

**RHS** counts number of binary strings of length n with n-k ones and k zeros

Rosen p. 411
Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof 1: Use formula

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}
\]

Proof 2: Combinatorial interpretation?

- **LHS** counts number of binary strings of length \( n \) with \( k \) ones and \( n-k \) zeros
- **RHS** counts number of binary strings of length \( n \) with \( n-k \) ones and \( k \) zeros

Can match up these two sets by pairing each string with another where 0s, 1s are flipped. This bijection means the two sets have the same size. So LHS = RHS.
Theorem: \[ \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \]

Proof 1: Use formula

Proof 2: Combinatorial interpretation?

LHS counts number of binary strings ???
RHS counts number of binary strings ???
Theorem:

\[
\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}
\]

Proof 2: Combinatorial interpretation?

**LHS** counts number of binary strings of length n+1 that have k ones.

**RHS** counts number of binary strings ***???***
Theorem: \[
\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}
\]

Proof 2: Combinatorial interpretation?

LHS counts number of binary strings of length n+1 that have k ones.
RHS counts number of binary strings ???
How many length $n+1$ strings start with 1 and have $k$ ones in total?

A. $C(n+1, k+1)$
B. $C(n, k)$
C. $C(n, k+1)$
D. $C(n, k-1)$
E. None of the above.
How many length n+1 strings start with 0 and have k ones in total?

A. $C(n+1, k+1)$
B. $C(n, k)$
C. $C(n, k+1)$
D. $C(n, k-1)$
E. None of the above.
Pascal's Identity

Theorem: \[ \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \]

Proof 2: Combinatorial interpretation?

**LHS** counts number of binary strings of length n+1 that have k ones.

**RHS** counts number of binary strings of length n+1 that have k ones, split in two.

Rosen p. 418
Theorem: \[ \sum_{k=0}^{n} \binom{n}{k} = 2^n \]

What set does the **LHS** count?

A. Binary strings of length n that have k ones.
B. Binary strings of length n that start with 1.
C. Binary strings of length n that have any number of ones.
D. None of the above.

Rosen p. 417
Theorem: \[ \sum_{k=0}^{n} \binom{n}{k} = 2^n \]

**Proof**: Combinatorial interpretation?

**LHS** counts number of binary strings of length \( n \) that have any number of 1s.

By sum rule, we can break up the set of binary strings of length \( n \) into disjoint sets based on how many 1s they have, then add their sizes.

**RHS** counts number of binary strings of length \( n \).

This is the same set so **LHS** = **RHS**.

Rosen p. 417
Announcements

Office Hours
Mine are Friday 10-11 and Saturday 3:30-5:30.
Lots on the course calendar!

HW6
Due Sunday 10pm