Tartaglia's Pouring Problem

How many configurations are possible?

A. Infinitely many
B. $4 \times 6 \times 9 = 216$
C. 24
D. 16
E. None of the above.

(l, m, s) means

l ounces in large cup
m ounces in medium cup
s ounces in small cup
Tartaglia's Pouring Problem

How many configurations are possible?

Small cup: 0, 1, 2, or 3
Medium cup: 0, 1, 2, 3, 4, or 5
Large cup: 0, 1, 2, 3, 4, 5, 6, 7, or 8

(l, m, s)**

means

l ounces in large cup
m ounces in medium cup
s ounces in small cup

**integer values
Tartaglia's Pouring Problem

How many configurations are possible?

Small cup: 0, 1, 2, or 3
Medium cup: 0, 1, 2, 3, 4, or 5
Large cup: 0, 1, 2, 3, 4, 5, 6, 7, or 8

But can't have 3 in small AND 5 in medium AND 8 in large: Total must be 8.

(l, m, s)** means
l ounces in large cup
m ounces in medium cup
s ounces in small cup

**integer values
Tartaglia's Pouring Problem

(l, m, s)** means
l ounces in large cup
m ounces in medium cup
s ounces in small cup

**integer values

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The three columns total 8 in each row. 24 rows.
Which configurations are actually possible?

Construct a graph by considering all possible moves (pours) from each configuration.
Tartaglia's Pouring Problem

Which configurations are *actually* possible?

Construct a graph by considering all possible moves (pours) from each configuration.
Tartaglia's Pouring Problem

Which configurations are actually possible?

Construct a graph by considering all possible moves (pours) from each configuration.
Tartaglia's Pouring Problem

Which configurations are *actually* possible?

Construct a graph by considering all possible moves (pours) from each configuration.
Tartaglia's Pouring Problem

Which configurations are *actually* possible?

Possible in blue.  Impossible in red.

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Graph reachability: WHAT

Given a directed graph $G$ and a start vertex $s$,

produce a list of all vertices $v$ reachable from $s$ by a directed path in $G$. 
Graph reachability: HOW

Given a directed graph $G$ and a start vertex $s$,

produce a list of all vertices $v$ reachable from $s$ by a directed path in $G$.

At each point in a graph search algorithm, the vertices are partitioned into

$X$: eXplored

$F$: frontier (reached but haven't yet explored)

$U$: unreached
procedure GraphSearch (G: directed graph, s: vertex)

Initialize $X = \text{empty}, F = \{s\}, U = V - F$.
While $F$ is not empty:
  Pick $v$ in $F$.
  For each neighbor $u$ of $v$:
    If $u$ is not in $X$ or $F$, then move $u$ from $U$ to $F$.
  Move $v$ from $F$ to $X$.

Return $X$. 
procedure **GraphSearch** (G: directed graph, s: vertex)

Initialize $X = \emptyset$, $F = \{s\}$, $U = V - F$.
While $F$ is not empty:
  Pick $v$ in $F$.
  For each neighbor $u$ of $v$:
    If $u$ is not in $X$ or $F$, then move $u$ from $U$ to $F$.
  Move $v$ from $F$ to $X$.

Return $X$.

**Before any iterations of while loop…**

$X = \emptyset$  
$F = \{(8,0,0)\}$  
$U =$ green nodes
procedure GraphSearch (G: directed graph, s: vertex)

Initialize X = empty, F = \{s\}, U = V – F.
While F is not empty:
Pick v in F.
For each neighbor u of v:
  If u is not in X or F, then move u from U to F.
Move v from F to X.
Return X.

After first iteration of while loop…

v = (8,0,0)
X = \{(8,0,0)\}  F = \{(3,5,0), (5,0,3)\}  U = green nodes
Different methods of choosing v give breadth-first search vs. depth-first search

**procedure GraphSearch** (G: directed graph, s: vertex)

Initialize X = empty, F = {s}, U = V − F.
While F is not empty:
  Pick v in F.
  For each neighbor u of v:
    If u is not in X or F, then move u from U to F.
  Move v from F to X.

Return X.

*After second iteration of while loop…*

v = (8,0,0)
X = {(8,0,0), (3,5,0)}  F = {(3,2,3), (0,5,3), (5,0,3)}
Graph search: WHY

**procedure** GraphSearch *(G: directed graph, s: vertex)*

Initialize X = empty, F = {s}, U = V – F.
While F is not empty:
  Pick v in F.
  For each neighbor u of v:
    If u is not in X or F, then move u from U to F.
  Move v from F to X.

Return X.

*Does this algorithm output the collection of vertices v reachable from s by a directed path in G?*
Graph reachability: WHY

procedure GraphSearch (G: directed graph, s: vertex)

Initialize X= empty, F = {s}, U = V – F.
While F is not empty:
   Pick v in F.
   For each neighbor u of v:
      If u is not in X or F, then move u from U to F.
   Move v from F to X.

Return X.

Does this algorithm output the collection of vertices v reachable from s by a directed path in G?

Goal:
1. Every element of output X is reachable from s in G.
2. Every reachable vertex is in X (by end of algorithm).
Graph reachability: WHY

procedure GraphSearch (G: directed graph, s: vertex)

Initialize X= empty, F = \{s\}, U = V – F.
While F is not empty:
   Pick v in F.
   For each neighbor u of v:
      If u is not in X or F, then move u from U to F.
   Move v from F to X.

Return X.

Claim: After \(t^{th}\) iteration through while loop, every element of (current version of) X or F is reachable from s in G.

Proof by induction on \(t\).
Graph reachability: WHY

procedure GraphSearch (G: directed graph, s: vertex)

Initialize X = empty, F = {s}, U = V – F.
While F is not empty:
Pick v in F.
For each neighbor u of v:
    If u is not in X or F, then move u from U to F.
Move v from F to X.
Return X.

Claim: After t^{th} iteration through while loop, every element of (current version of) X or F is reachable from s in G.

Base case (t=0):
Before any iterations of loop, X is initialized as empty and F is initialized as {s}. WTS s is reachable from s in G. ☺
procedure GraphSearch (G: directed graph, s: vertex)

Initialize X= empty, F = {s}, U = V – F.
While F is not empty:
  Pick v in F.
  For each neighbor u of v:
    If u is not in X or F, then move u from U to F.
  Move v from F to X.

Return X.

Claim: After $t^{th}$ iteration through while loop, every element of (current version of) X or F is reachable from s in G.

Induction step: Suppose after $t^{th}$ iteration, every element of X or F is reachable from s in G.

What happens in $t+1^{st}$ iteration?
**Procedure GraphSearch** (G: directed graph, s: vertex)

Initialize X = empty, F = {s}, U = V – F.
While F is not empty:
- Pick v in F.
- For each neighbor u of v:
  - If u is not in X or F, then move u from U to F.
- Move v from F to X.

Return X.

**Claim:** After $t^{th}$ iteration through while loop, every element of (current version of) X or F is reachable from s in G.

**Induction step:** Suppose that after $t^{th}$ iteration, every element of X or F is reachable from s in G.

What happens in $t+1^{st}$ iteration?
procedure GraphSearch (G: directed graph, s: vertex)

Initialize X = empty, F = \{s\}, U = V – F.
While F is not empty:
    Pick v in F.
    For each neighbor u of v:
        If u is not in X or F, then move u from U to F.
    Move v from F to X.

Return X.

Claim: After t^{th} iteration through while loop, every element of (current version of) X or F is reachable from s in G.

Using Claim to prove Goal 1: After the final iteration, output X, which by claim, only contains vertices that are reachable from s.
procedure GraphSearch (G: directed graph, s: vertex)

Initialize X= empty, F = {s}, U = V – F.
While F is not empty:
   Pick v in F.
   For each neighbor u of v:
      If u is not in X or F, then move u from U to F.
   Move v from F to X.

Return X.

Does this algorithm output the collection of vertices v reachable from s by a directed path in G?

Goal:
1. Every element of output X is reachable from s in G. ☺
2. Every reachable vertex is in X (by end of algorithm).
**Graph reachability: WHY**

**procedure GraphSearch** (G: directed graph, s: vertex)

Initialize X = empty, F = \{s\}, U = V – F.

While F is not empty:
  - Pick v in F.
  - For each neighbor u of v:
    - If u is not in X or F, then move u from U to F.
  - Move v from F to X.

Return X.

**WTS Goal 2: Every reachable vertex is in X.**

*Hint: assume, towards a contradiction that some vertex is reachable from s but not in X. Look for first vertex on the path between s that is not in X.*
Graph reachability: WHEN

procedure **GraphSearch** (G: directed graph, s: vertex)

Initialize X= empty, F = \{s\}, U = V – F.
While F is not empty:
    Pick v in F.
    For each neighbor u of v:
        If u is not in X or F, then move u from U to F.
    Move v from F to X.

Return X.

How long does it take to pick v in F?
How long does it take to iterate over neighbors of v?

Need to know some implementation decisions.
Graph reachability: WHEN

**procedure** GraphSearch (G: directed graph, s: vertex)

Initialize X = empty, F = {s}, U = V – F.
While F is not empty:
    Pick v in F.
    For each neighbor u of v:
        If u is not in X or F, then move u from U to F.
    Move v from F to X.
Return X.

What's an upper bound on the time it takes to do one iteration of the body of the for loop?

A. O( n^2 )
B. O(n)
C. O( degree (v) )
D. O( |F| )
E. None of the above.

Assume G stored as adjacency list.
Assume have array Status[]
* length n array
* each entry either F, X, U
Procedure GraphSearch (G: directed graph, s: vertex)

Initialize X = empty, F = {s}, U = V – F.
While F is not empty:
    Pick v in F.
    For each neighbor u of v:
        If u is not in X or F, then move u from U to F.
    Move v from F to X.
Return X.

Assume G stored as adjacency list.
Assume have array Status[]
    * length n array
    * each entry either F, X, U

What's an upper bound on the time it takes to go through the whole for loop for a given v?

A. O( n^2 )
B. O(n)
C. O( degree (v) )
D. O(|F|)
E. None of the above.
procedure GraphSearch \((G: \text{directed graph}, \ s: \text{vertex})\)

Initialize \(X = \text{empty}, \ F = \{s\}, \ U = V - F\).
While \(F\) is not empty:
  Pick \(v\) in \(F\).
  For each neighbor \(u\) of \(v\):
    If \(u\) is not in \(X\) or \(F\), then move \(u\) from \(U\) to \(F\).
    Move \(v\) from \(F\) to \(X\).

Return \(X\).

Assume \(G\) stored as adjacency list.
Assume have array \(\text{Status}[]\)
* length \(n\) array
  * each entry either \(F\), \(X\), \(U\)

What's an upper bound on the time spent on the for loop throughout the whole algorithm?

A. \(O( n )\)
B. \(O( |V| )\)
C. \(O( |E| )\)
D. \(O( |F| )\)
E. None of the above.
procedure GraphSearch (G: directed graph, s: vertex)

Initialize X= empty, F = {s}, U = V – F.
While F is not empty:
    Pick v in F.
    For each neighbor u of v:
        If u is not in X or F, then move u from U to F.
    Move v from F to X.

Return X.

Assume G stored as adjacency list.
Assume have array Status[]
  * length n array
  * each entry either F, X, U

Total time is asymptotically upper bounded by sum of degrees of all vertices

i.e. O (2 |E| )

i.e. O( |E| )
Another Special Type of Graph: Trees
1. Definitions of trees

2. Properties of trees

3. Revisiting uses of trees
A **rooted tree** is a connected directed acyclic graph in which one vertex has been designated the root, which has no incoming edges, and every other vertex has exactly one incoming edge.
A **rooted tree** is a connected directed acyclic graph in which one vertex has been designated the root, which has no incoming edges, and every other vertex has exactly one incoming edge.

Special case of DAGs from last class.
Note that each vertex in middle has *exactly one* incoming edge from layer above.
Edges are directed *away from* the root.
Which of the following directed graphs are trees (with root indicated in green)?

A. 

B. 

C. 

D.
(Rooted) Trees: definitions

Rosen p. 747-749
If vertex $v$ is not the root, it has exactly one incoming edge, which is from its parent, $p(v)$.

**Height** of vertex $v$ is given by the recurrence:

$$h(v) = h(p(v)) + 1 \quad \text{if } v \text{ is not the root, and}$$

$$h(r) = 0$$
Height of vertex \( v \):
\[
h(v) = h(p(v)) + 1 \quad \text{if } v \text{ is not the root, and} \quad h(r) = 0
\]

What is the height of the red vertex?

A. 0
B. 1
C. 2
D. 3
E. None of the above.
(Rooted) Trees: definitions

**Height** of vertex $v$: $h(v) = h(\ p(v)\ ) + 1$ \textit{if v is not the root, and} $h(r) = 0$

**Height** of tree is maximum height of a vertex in the tree.

Rosen p. 753
A binary tree is a rooted tree where every (internal) vertex has no more than 2 children.

How many leaves does a binary tree of height 3 have?
A. 2  
B. 3  
C. 6  
D. 8  
E. None of the above.
A binary tree is a rooted tree where every (internal) vertex has no more than 2 children.

How many leaves does a binary tree of height 3 have?
A. 2
B. 3
C. 6
D. 8
E. None of the above.

*See Theorem 5 for proof of upper bound*
A **full** binary tree is a rooted tree where every internal vertex has exactly 2 children.

Which of the following are full binary trees?

A.  
B.  
C.  
D.
A full binary tree is a rooted tree where every internal vertex has exactly 2 children.

At most how many vertices are there in a full binary tree of height $h$?

A. $\Theta(h)$

B. $\Theta(2^h)$

C. $\Theta(h^2)$

D. $\Theta(\log h)$

Max number of vertices when tree is balanced
A **full** binary tree is a rooted tree where every internal vertex has exactly 2 children.

**Key insight:** number of vertices doubles on each level.

\[
1 + 2 + 4 + 8 + \ldots + 2^h = 2^{h+1} - 1 \quad \text{i.e.} \quad \Theta(2^h)
\]

If \( n \) is number of vertices:

\[
n = 2^{h+1} - 1
\]

so

\[
h = \log(n+1) - 1 \quad \text{i.e.} \quad \Theta(\log n)
\]
Binary tree

Relating height and number of vertices:

\[ \log(n+1) - 1 \leq h \leq ___ \]

This is what we just proved.

How do we prove?

What tree with \( n \) vertices has the greatest possible height?
Relating height and number of vertices:

\[ \log(n+1) - 1 \leq h \leq n - 1 \]

This is what we just proved.

How do we prove?

What tree with \( n \) vertices has the greatest possible height?
Trees

1. Definitions of trees
2. Properties of trees
3. Revisiting uses of trees

In data structures:
Binary search trees
Binary Search Trees

- Facilitate binary search (must maintain sorted order of data)
- Dynamic

**Implementation**

Each vertex is an object with the fields

\[ p = \text{parent} \]
\[ lc = \text{left child} \]
\[ rc = \text{right child} \]
\[ \text{value} \]

**When is \( p \) null?**

- A. If we have an error in our implementation.
- B. When the value is 0.
- C. When the vertex is a leaf node.
- D. When the vertex is the root node.
- E. None of the above.
Binary Search Trees

- Facilitate binary search (must maintain sorted order of data)
- Dynamic

Implementation

Each vertex is an object with the fields

\[ \begin{align*}
\text{p} & = \text{parent} \\
\text{lc} & = \text{left child} \\
\text{rc} & = \text{right child} \\
\text{value} & \\
\end{align*} \]

When is \texttt{lc null}? 

A. If we have an error in our implementation.
B. When the value is 0.
C. When the vertex is a leaf node.
D. When the vertex is the root node.
E. None of the above.
Binary Search Trees

- Facilitate binary search (must **maintain sorted order** of data)
- Dynamic

For each vertex v
- If x is in subtree rooted at lc(v), \( \text{value}(x) \leq \text{value}(v) \).
- If x is in the subtree rooted at rc(v), \( \text{value}(x) \geq \text{value}(v) \).
Binary Search Trees

- Facilitate binary search (must maintain sorted order of data)
- Dynamic

How would you search for "orange?"
Binary Search Trees

- Facilitate binary search (must maintain sorted order of data)
- Dynamic

To search for target T in a binary search tree.

1. Compare T to value(r) where r is the root.
2. If T = value(r), done 😊.
3. If T < value(r), search recursively starting at lc(r).
4. If T > value(r), search recursively starting at rc(r).
Binary Search Trees

To search for target $T$ in a binary search tree.

1. Compare $T$ to $\text{value}(r)$ where $r$ is the root.
2. If $T = \text{value}(r)$, done 😊.
3. If $T < \text{value}(r)$, search recursively starting at $\text{lc}(r)$.
4. If $T > \text{value}(r)$, search recursively starting at $\text{rc}(r)$.

How long does this take?
Binary Search Trees

- Facilitate binary search (must maintain sorted order of data)
- Dynamic

To search for target $T$ in a binary search tree.

1. Compare $T$ to $\text{value}(r)$ where $r$ is the root.
2. If $T = \text{value}(r)$, done 😊.
3. If $T < \text{value}(r)$, search recursively starting at $\text{l}c(r)$.
4. If $T > \text{value}(r)$, search recursively starting at $\text{r}c(r)$.

**How long does this take?** Constant time at each level, number of levels is height+1.
Binary Search Trees

- Facilitate binary search (must **maintain sorted order** of data)
- Dynamic

To search for target T in a binary search tree.

1. Compare T to \( \text{value}(r) \) where \( r \) is the root.
2. If \( T = \text{value}(r) \), done 😊.
3. If \( T < \text{value}(r) \), search recursively starting at \( \text{lc}(r) \).
4. If \( T > \text{value}(r) \), search recursively starting at \( \text{rc}(r) \).

**How long does this take?** Time proportional to height!
An **unrooted tree** is a connected undirected graph with no cycles.

Rosen p. 746
Theorem: An undirected graph is an unrooted tree if and only if it contains all the edges of some rooted tree.

What does this mean?

(1) If we replace all directed edges in a rooted tree with undirected edges, the result will be an unrooted tree.

(2) There is always some way to put directions on the edges of an unrooted tree to make it a rooted tree.
Goal (1): If we replace all directed edges in a rooted tree with undirected edges, the result will be an unrooted tree.

What do we need to prove?

A. The resulting undirected graph will be connected.
B. The resulting undirected graph will be undirected.
C. The resulting undirected graph will not have cycles.
D. All of the above.
**Goal (1):** If we replace all directed edges in a rooted tree with undirected edges, the result will be an **unrooted tree**.

**SubGoal (1a):** this resulting graph is connected, i.e. between any two vertices $u$ and $v$ there is a path in the graph.

*Idea:* To find path between purple and orange, follow parents of purple all the way to root, then follow its children down to orange.
Equivalence between rooted and unrooted trees

**Goal (1):** If we replace all directed edges in a rooted tree with undirected edges, the result will be an unrooted tree.

**SubGoal (1b):** this resulting graph has no cycles.

**Idea:** Towards a contradiction, assume there is a cycle and consider the simplest cycle (with no repeated vertices). Start at vertex at highest level in the cycle. Next step must go to a child node, etc. Can never go up to higher level again because vertices in rooted tree only have one incoming edge.
Goal (2): There is always some way to put directions on the edges of an unrooted tree to make it a rooted tree.

Idea: finding right directions for edges will be similar to finding topological order of a DAG.
**Goal (2):** There is always some way to put directions on the edges of an unrooted tree to make it a rooted tree.

**SubGoal (2a):** Any unrooted tree with at least two vertices has a vertex of degree exactly 1.

**Proof:** Towards a contradiction, assume that all vertices have degree 0 or $\geq 2$. Since a tree is connected, eliminate the case of degree-0 vertices. **Goal:** construct a cycle to arrive at a contradiction.

Start at any vertex $u_0$. Pick $u_{i+1}$ so that it is adjacent to $u_i$ but is not $u_{i-1}$. **Why?**

Get $u_0, u_1, \ldots, u_n$. By Pigeonhole Principle, must repeat. **Cycle!**
**Goal (2):** There is always some way to put directions on the edges of an unrooted tree to make it a rooted tree.

**SubGoal (2b):** If $T$ is unrooted tree and $v$ has degree 1 in $T$, then $T\{v\}$ is unrooted tree.

**Proof:** To check that $T\{v\}$ is unrooted tree,

* confirm $T\{v\}$ is connected and

* $T\{v\}$ does not have a cycle.
**Goal (2):** There is always some way to put directions on the edges of an unrooted tree to make it a rooted tree.

**SubGoal (2b):** If $T$ is unrooted tree and $v$ has degree 1 in $T$, then $T-\{v\}$ is unrooted tree.

**Proof:** To check that $T-\{v\}$ is unrooted tree,

* confirm $T-\{v\}$ is connected and

* $T-\{v\}$ does not have a cycle.
Equivalence between rooted and unrooted trees

**Goal (2):** There is always some way to put directions on the edges of an unrooted tree to make it a rooted tree.

Using the subgoals to achieve the goal:

Root($T$: unrooted tree with $n$ nodes)
1. If $n=1$, let the only vertex $v$ be the root, set $h(v):=0$, and return.
2. Find a vertex $v$ of degree 1 in $T$, and let $u$ be its only neighbor.
3. Root($T$-$\{v\}$).
4. Set $p(v):=u$ and $h(v):=h(u)+1$. 

Recursion!
Announcements

HW5 Due Tuesday 10pm

Office Hours

Mine are Friday 10-11, Saturday 1-3, CSE 4204.

Lots more on the course calendar!