Eulerian Tours and Fleury’s Algorithm

CSE21 Fall 2016, Day 12

October 21, 2016

http://cseweb.ucsd.edu/classes/fa16/cse21-ab/
Vocabulary

Path (or walk): describes a route from one vertex to another 
\((v_1, e_1, v_2, e_2, ..., v_k)\)

Length of path: number of edges

Simple path: path that doesn’t repeat vertices

Circuit (or cycle or closed walk): path that starts and ends at the same vertex, with length greater than zero

Loop (or self-loop): an edge from a vertex to itself
Consider only **undirected** graphs.

**1\textsuperscript{st} goal:** Determine whether a given undirected graph $G$ has an Eulerian tour.

**2\textsuperscript{nd} goal:** Actually find an Eulerian tour in an undirected graph $G$, when possible.

Path that includes each edge once
Finding Eulerian tours

How many paths are there between vertex A and vertex B?

A. None.
B. Exactly one.
C. Exactly two.
D. More than two.
E. None of the above.
Finding Eulerian tours

How many paths are there between vertex A and vertex I?

A. None.
B. Exactly one.
C. Exactly two.
D. More than two.
E. None of the above.
An undirected graph $G$ is **connected** if *for any* ordered pair of vertices $(v,w)$ there is a path from $v$ to $w$. 

*Connected*  

*Not connected*
An undirected graph $G$ is **connected** if for any ordered pair of vertices $(v, w)$ there is a path from $v$ to $w$.

An undirected graph $G$ is **disconnected** if

A. for any ordered pair of vertices $(v, w)$ there is no path from $v$ to $w$.  
B. there is an ordered pair of vertices $(v, w)$ with a path from $v$ to $w$.  
C. there is an ordered pair of vertices $(v, w)$ with no path from $v$ to $w$.  
D. for every ordered pair of vertices $(v, w)$ there is a path from $v$ to $w$.  
E. None of the above.
Disconnected graphs can be broken up into pieces where each is connected.

Each connected piece of the graph is a **connected component**.

Does this graph have an Eulerian tour?
Finding Eulerian tours

Let $G = (V,E)$ be an
- undirected
- connected
graph with $n$ vertices.

1st goal: Determine whether $G$ has an Eulerian tour.

2nd goal: If yes, find the tour itself.
Finding Eulerian tours

Observation:

If $v$ is an intermediate* vertex on a path $p$, then $p$ must enter $v$ the same number of times it leaves $v$.

* not the start vertex, not the end vertex.
Finding Eulerian tours

Observation:

If \( v \) is an \textbf{intermediate}\(^*\) vertex on a path \( p \), then \( p \) must \textbf{enter} \( v \) the same number of times it \textbf{leaves} \( v \).

An Eulerian tour contains all edges incident on \( v \).
So, half of the edges are used to enter \( v \), half to leave.

\(^*\) not the start vertex, not the end vertex.
Recall: Degree

The **degree** of a vertex in an undirected graph is the total number of edges **incident** with it, except that a loop contributes twice.

Rosen p. 652
Finding Eulerian tours

Observation:

If $v$ is an intermediate* vertex on a path $p$, then $p$ must enter $v$ the same number of times it leaves $v$.

An Eulerian tour contains all edges incident on $v$. So, half of the edges are used to enter $v$, half to leave.

* not the start vertex, not the end vertex.
Finding Eulerian tours

(Summary of) Observation:

In an Eulerian tour, any intermediate vertex has even degree.

In a circuit, all vertices are intermediate so all have even degree.

In a tour starting and ending at different vertices (not a circuit), the starting and ending vertices will have odd degree, all others will have even degree.
Finding Eulerian tours

**Theorem:** If $G$ has an Eulerian tour, $G$ has at most two odd degree vertices.

Which of the following statements is **equivalent** to the theorem (using the facts we know so far about graphs) ?

A. If the number of odd degree vertices in $G$ is anything other than 0 or 2, then $G$ has no Eulerian tour.
B. If $G$ has three or more odd degree vertices, then $G$ does not have an Eulerian tour.
C. If $G$ has an Eulerian tour, then $G$ has either all vertices with even degree or $G$ has exactly two vertices with odd degree.
D. All of the above.
E. None of the above.
Theorem: If G has an Eulerian tour, G has at most two odd degree vertices.

Which of the following statements is the **converse** to the theorem?

A. If G does not have an Eulerian tour, then G does not have at most two odd degree vertices.
B. If G has at most two odd degree vertices, then G has an Eulerian tour.
C. If G does not have at most two odd degree vertices, then G does not have an Eulerian tour.
D. More than one of the above.
E. None of the above.
Finding Eulerian tours

**Theorem:** If G has an Eulerian tour, G has at most two odd degree vertices.

**Question:** is the converse also true? i.e

If G has at most two odd degree vertices, then must G have an Eulerian tour?
Finding Eulerian tours

**Theorem:** If G has an Eulerian tour, G has at most two odd degree vertices.

**Question:** is the converse also true? i.e

If G has at most two odd degree vertices, then must G have an Eulerian tour?

**Answer:** give algorithm to build the Eulerian tour!

*We'll develop some more graph theory notions along the way.*
Finding Eulerian tours

Eulerian tour?
Eulerian tour?

Start at 4. Where should we go next?

A. 2 or 3.
B. 2 or 5.
C. 3 or 5.
D. 2 or 3 or 5.
A **bridge** is an edge, which, if removed, would cause G to be disconnected.

Which of the edges in this graph are bridges?

A. D, E  
B. C, D  
C. D only  
D. None of the above.
A **bridge** is an edge, which, if removed, would cause $G$ to be disconnected.

**Connection with Eulerian tours:**

In an Eulerian tour, we have to visit **every edge** on one side of the bridge before we cross it (because there's no coming back).

*Do you see divide & conquer in here?*
Eulerian Tours: HOW (Fleury's Algorithm)

1. Check that G has at most 2 odd degree vertices.
2. Start at vertex v, an odd degree vertex if possible.
3. While there are still edges in G,
4. If there is more than one edge incident on v
5. Cross any edge incident on v that is not a bridge
6. Else, cross the only edge available from v.
7. Delete the edge just crossed from G, update v.
Eulerian Tours: HOW (Fleury's Algorithm)

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What Eulerian tour do you get when following Fleury's algorithm, starting at vertex 4?
Eulerian Tours: WHY (Fleury's Algorithm)

1. Check that G has at most 2 odd degree vertices.
2. Start at vertex v, an odd degree vertex if possible.
3. While there are still edges in G,
4. If there is more than one edge incident with v
5. Cross any edge incident with v that is not a bridge
6. Else, cross the only edge available from v.
7. Delete the edge just crossed from G, update v.

Will there always be such an edge?

Will go through each edge at most once, so if while loop iterates |E| times, get an Eulerian tour.
Eulerian Tours: WHY (Fleury's Algorithm)

1. Check that G has at most 2 odd degree vertices.
2. Start at vertex v, an odd degree vertex if possible.
3. While there are still edges in G,
   4. If there is more than one edge incident with v
   5. Cross any edge incident with v that is not a bridge
   6. Else, cross the only edge available from v.
   7. Delete the edge just crossed from G, update v.

Need to show (loop invariant): Connected graph with no more than one bridge from v.

If we enter the while loop, then after $t^{th}$ iteration of while loop,
- the (remaining) graph is still connected, AND
- there is at most one other odd degree vertex in the (remaining) graph other than v, AND
- across every bridge from v there is an odd degree vertex (and hence there is at most one bridge incident with v).
Eulerian Tours WHY Fleury's Algorithm

1. Check that G has at most 2 odd degree vertices.
2. Start at vertex v, an odd degree vertex if possible.
3. While there are still edges in G,
   4. If there is more than one edge incident with v
   5. Cross any edge incident with v that is not a bridge
   6. Else, cross the only edge available from v.
   7. Delete the edge just crossed from G, update v.

Will there always be such an edge?

Why is more than one bridge from v bad?
Seven Bridges of Konigsberg

Is there a path where each edge occurs exactly once? **Eulerian tour**

No!

Rosen p. 693
Seven Bridges of Konigsberg redux

Which of these puzzles can you draw without lifting your pencil off the paper?
A. No
B. No
C. No
D. Yes
Consider only **undirected** graphs.

1\(^{st}\) goal: Determine whether a given undirected graph \(G\) has an Eulerian tour. **\(G\) has an Eulerian tour if and only if \(G\) has at most 2 odd-degree vertices.**

2\(^{nd}\) goal: Actually find an Eulerian tour in an undirected graph \(G\), when possible. **Fleury's Algorithm: don't burn your bridges.**
### Eulerian Tours: recap

<table>
<thead>
<tr>
<th>Number of odd degree vertices</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3,5,7,...</th>
<th>4,6,8,...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Is such a graph possible?</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
</tr>
<tr>
<td>Is an Eulerian tour possible?</td>
<td>yes, a circuit</td>
<td>yes, not a circuit</td>
<td>no</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Announcements

Office Hours
Today 10-11, Tomorrow 4-6
Lots of TA and tutor office hours today and Sunday.

Today is deadline to drop without W

HW4 Due Tuesday