1. Recall that a function \( f : X \to Y \) is
   (i) \textit{injective} if for all \( x, x' \in X \), \( f(x) = f(x') \implies x = x' \),
   (ii) \textit{surjective} if for all \( y \in Y \), there exists \( x \in X \) such that \( f(x) = y \), and
   (iii) \textit{bijective} if it is both injective and surjective.

For the following functions, determine if they are injective, surjective, or bijective. Prove your answer. If you claim that a function is only injective, you must prove that it is injective and not surjective. Similarly, if you claim a function is only surjective, you must prove it is surjective and not injective.

(a) Define \( f : \mathbb{Z} \to \mathbb{Z} \) such that \( f(x) = 3x \).

(b) Define \( g : \mathbb{N} \to \mathbb{N} \cup \{0\} \) such that \( g(x) = \lfloor x / 2 \rfloor \). (For a real number \( a \in \mathbb{R} \), \( \lfloor a \rfloor \) is the largest integer \( z \) such that \( z \leq a \). You may use the fact that if \( a = z + r \) where \( z \in \mathbb{Z} \) and \( 0 \leq r < 1 \), then \( \lfloor a \rfloor = z \).

2. Recall that for sets \( A \) and \( B \), the \textit{power set} of \( A \) is given by \( P(A) = \{ X \mid X \subseteq A \} \) and the \textit{cross product} of \( A \) and \( B \) is given by \( A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \} \).

Finally, given \((a, b), (a', b') \in A \times B\), we have \((a, b) = (a', b')\) if and only if \( a = a' \) and \( b = b' \). For two sets \( A \) and \( B \), define the function \( f_{A,B} : P(A \cup B) \to P(A) \times P(B) \) as
   \[ f_{A,B}(X) = (X \cap A, X \cap B). \]

(a) Is \( f_{A,B} \) injective, surjective, or bijective? Prove your answer.

(b) Suppose now that \( A \cap B = \emptyset \). Is \( f_{A,B} \) injective, surjective, or bijective? Prove your answer.

\textit{Hint: you may find it useful to prove the following lemma: suppose } \( X \subseteq Y \), \textit{then } \( X \cap Y = X \).

3. Recall that a relation is an equivalence relation if it is reflexive, symmetric, and transitive. For the following relations, determine whether or not they are equivalence relations. If they are equivalence relations, prove that they satisfy all three properties. If they are not, find the property that they do not satisfy and provide a counter example.

(a) Let \( R_1 \) be the relation over \( \mathbb{Q} \) such that \( a R_1 b \) if and only if \( a - b \in \mathbb{Z} \).

(b) Let \( R_2 \) be the relation over \( \mathbb{N} \times \mathbb{N} \) such that \( (a, b) R_2 (c, d) \) if and only if \( ad = bc \). (Here, we say \( \mathbb{N} \) is the set of \textit{positive} integers, not including 0).

(c) Let \( R_3 \) be the relation over \( \mathbb{Z} \) such that \( a R_3 b \) if and only if \( |a - b| \leq 5 \).

*(Bonus points). For each of the above relations you have shown to be equivalence relations, identify the corresponding equivalence classes. Explain your reasoning.

4. Recall that a set \( A \) is called \textit{countable} if
   (i) \( A \) is infinite and
(ii) there exists a surjective function $g : \mathbb{N} \to A$.

Prove that if $A$ is a countable set and $B$ is a countable set then $A \cup B$ is countable.

*5. (Bonus question) Suppose for each $n \in \mathbb{N}$ there exists a countable set $S_n$. Show that the union of all such $S_n$, written

$$S = \bigcup_{n=1}^{\infty} S_n,$$

is countable.

*Hint: From the countability proof of the rationals in lecture, we actually saw that the set $\mathbb{N} \times \mathbb{N}$ is countable. You may use the fact that $\mathbb{N} \times \mathbb{N}$ is countable in your proof.