Reminders

NO CONVERSATIONS about exam until Friday at 11am

Discussion section tomorrow: go over solutions of exam.
Today's learning goals

• Justify why the Pumping Lemma is true
• Apply the Pumping Lemma in proofs of nonregularity
• Identify some nonregular sets
Regular languages

To prove that a set of strings over the alphabet $\Sigma$ is regular,

- Build a **DFA** whose language is this set.
- Build an **NFA** whose language is this set.
- Use the **closure properties** of the class of regular languages to construct this set from others known to be regular.
  - Union
  - Intersection
  - Complementation
  - Concatenation
  - Flip bits
  - Kleene star
Where we stand

• There exist non-regular sets.

• If we know that some sets are not regular, we can conclude others are also not regular judiciously reasoning using closure properties of class of regular languages.

• No example of a specific regular set ... yet.
Bounds on DFA

• in DFA, memory = states

• Automata can only "remember"…
  • …finitely far in the past
  • …finitely much information

• If a computation path visits the same state more than once, the machine can't tell the difference between the first time and future times it visited that state.
Example!

\[ \{ 0^n1^n \mid n \geq 0 \} \]

What are some strings in this set?
What are some strings not in this set?

Compare to \( L(0^*1^*) \)

Design a DFA? NFA?
Example!

\{ 0^n1^n \mid n \geq 0 \}

What are some strings in this set?

What are some strings not in this set?

Compare to \( L(0^*1^*) \)

Design a DFA? NFA?
Pumping

- Focus on computation path through DFA
Pumping

- Focus on computation path through DFA
Pumping

- Focus on computation path through DFA

Idea: if one long string is accepted, then many other strings have to be accepted too.
Pumping Lemma

If $A$ is a regular language, then there is a number $p$ (the pumping length) where, if $s$ is any string in $A$ of length at least $p$, then $s$ may be divided into three pieces, $s = xyz$ such that

- $|y| > 0$, and
- for each $i \geq 0$, $xy^iz \in A$,
- $|xy| \leq p$. 
Pumping Lemma

If \( A \) is a regular language, then there is a number \( p \) (the pumping length) where if \( s \) is any string in \( A \) of length at least \( p \), then \( s \) may be divided into three pieces, \( s = xyz \) such that

1. \( |y| > 0 \), and
2. for each \( i \geq 0 \), \( xy^iz \in A \),
3. \( |xy| \leq p \).
Using the Pumping Lemma

Claim: The set $L = \{0^n1^n \mid n \geq 0\}$ is not regular.

**Proof:** Assume $L$ is regular. So $L$ has pumping length $P$. 
Using the Pumping Lemma

Claim: The set $L = \{0^n1^n \mid n \geq 0\}$ is not regular.

Proof: Assume, towards a contradiction, that $L$ is regular.
Using the Pumping Lemma

Claim: The set \( L = \{0^n1^n \mid n \geq 0\} \) is not regular.

Proof: Assume, towards a contradiction, that \( L \) is regular. Therefore, the Pumping Lemma applies to \( L \) and gives us some number \( p \), the pumping length of \( L \). In particular, this means that every string in \( L \) that is of length \( p \) or more can be "pumped".

...Idea: can we find some long string in \( L \) that can't be?
Using the Pumping Lemma

Claim: The set $L = \{0^n1^n \mid n \geq 0\}$ is not regular.

Proof: …In particular, this means that every string in $L$ that is of length $p$ or more can be "pumped".

Goal: pick a string $s$ in $L$ of length greater than or equal to $p$ such that any division of $s$ as $s = xyz$ with $|y| > 0$ and $|xy| \leq p$ gives some value $i \geq 0$ with $xy^iz$ not in $L$. So we have a contradiction, and $L$ is not regular.
Using the Pumping Lemma

Claim: The set \( L = \{0^n1^n \mid n \geq 0\} \) is not regular.

Proof: …

Goal: pick a string \( s \) in \( L \) of length greater than or equal to \( p \) such that any division of \( s \) as \( s = xyz \) with \( |y| > 0 \) and \( |xy| \leq p \) gives some value \( i \geq 0 \) with \( xy^iz \) not in \( L \)

Choose \( s = 0^p1^p \). Consider any \( s = xyz \) with \( |y| > 0 \), \( |xy| \leq p \).
Claim: The set $L = \{0^n1^n \mid n \geq 0\}$ is not regular.

Goal: pick a string $s$ in $L$ of length greater than or equal to $p$ such that any division of $s$ as $s = xyz$ with $|y| > 0$ and $|xy| \leq p$ gives some value $i \geq 0$ with $xy^iz$ not in $L$.

Choose $s = 0^p1^p$. Consider any $s = xyz$ with $|y| > 0$, $|xy| \leq p$. Since $|xy| \leq p$, $x = 0^m$, $y = 0^n$, $z = 0^r1^p$ with $m + n + r = p$, $j > 0$. 
Using the Pumping Lemma

Claim: The set \( L = \{0^n1^n \mid n \geq 0\} \) is not regular.

Proof: ...

Goal: pick a string \( s \) in \( L \) of length greater than or equal to \( p \) such that any division of \( s \) as \( s = xyz \) with \(|y|>0\) and \(|xy|\leq p\) gives some value \( i \geq 0 \) with \( xy^iz \) not in \( L \).

Choose \( s = 0^p1^p \). Consider any \( s = xyz \) with \(|y|>0\), \(|xy|\leq p\).

Since \(|xv|\leq p\), \( x=0^m \), \( y=0^n \), \( z=0^r1^p \) with \( m+n+r=p \), \( j>0 \).

Picking \( i=0 \): \( xy^iz = xz = 0^m0^r1^p = 0^{m+r}1^p \), not in \( L \)!
Using the Pumping Lemma

Claim: The set \( L = \{0^n1^n \mid n \geq 0\} \) is not regular.

Proof: …

Goal: pick a string \( s \) in \( L \) of length greater than or equal to \( p \) such that any division of \( s \) as \( s = xyz \) with \(|y| > 0 \) and \(|xy| \leq p \) gives some value \( i \geq 0 \) with \( xy^iz \) not in \( L \)

Choose \( s = 0^p1^p \). Consider any \( s = xyz \) with \(|y| > 0 \), \(|xy| \leq p \).

Since \(|xy| \leq p \), \( x = 0^m \), \( y = 0^n \), \( z = 0^r1^p \) with \( m+n+r = p \), \( j > 0 \).

Picking \( i = 0 \): \( xy^iz = xz = 0^m0^r1^p = 0^{m+r}1^p \), not in \( L \)! This is a contradiction with the Pumping Lemma applied to \( L \), so \( L \) must not be regular.
Key ingredients in proof

**Claim**: Language L is not regular.

**Proof**: Assume, towards a contradiction, that L is regular. By the Pumping Lemma, there is a pumping length p for L. Consider the string $s = \ldots$ You must pick s carefully: we want $|s| \geq p$ and s in L. *Confirm these facts as part of your proof*

Now we will prove a contradiction with the statement "s can be pumped"

Consider an arbitrary choice of $x, y, z$ such that $s = xyz$, $|y| > 0$, $|xy| \leq p$. **This means that**... What properties are guaranteed about x, y, z?

Consider $i = \ldots$ In this case, $xy^iz = \ldots$, which is not in L, a contradiction with the Pumping Lemma applying to L and so L is not regular.
Using the Pumping Lemma

Claim: The set \( L = \{0^n1^n \mid n \geq 0\} \) is not regular.

In proof, we used \( s = 0^n1^n \) and \( i=0 \)

Claim: The set \( \{a^mb^ma^n \mid m,n \geq 0\} \) is not regular.

In proof, we used \( s = a^pb^ma^p \) and \( i=3 \)
Claim: The set \( \{ w w^R \mid w \text{ is a string over } \{0,1\} \} \) is not regular.

Proof: …Consider the string \( s = \ldots \) …

You must pick \( s \) carefully: we want \(|s| \geq p\) and \( s \) in \( L \). Now we will prove a contradiction with the statement "s can be pumped" Consider \( i=\ldots \)

Which \( s \) and \( i \) let us complete the proof?

A. \( s = 0^p0^p, \ i=2 \)  
B. \( s = 0110, \ i=0 \)  
C. \( s = 0^p110^p, \ i=1 \)  
D. \( s = 1^p001^p, \ i=3 \)  
E. I don't know
How do we choose \( i \)?

Claim: The set \( \{0^i1^j \mid i,j \geq 0 \text{ and } i \geq j \} \) is not regular.

Proof: …Consider the string \( s = \ldots \) …

You must pick \( s \) carefully: we want \(|s| \geq p\) and \( s \) in \( L \). Now we will prove a contradiction with the statement "\( s \) can be pumped" Consider \( i = \ldots \) …

Which \( s \) and \( i \) let us complete the proof?

A. \( s = 0^p1^p, i=2 \)  
B. \( s = 0^p1^p, i=p \)  
C. \( s = 0^p1^p, i=1 \)  
D. \( s = 0^p1^p, i=0 \)  
E. I don't know