Today's learning goals  Sipser Ch 1.1

• Design finite automata which accept a given language
• General Properties of Regular Languages
• Operations on languages
• Closure properties
The regular operations \textit{Sipser Def 1.23 p. 44}

For \(A, B\) languages over same alphabet, define:

\[ A \cup B = \{ x \mid x \in A \text{ or } x \in B \} \]
\[ A \circ B = \{ xy \mid x \in A \text{ and } y \in B \} \]
\[ A^* = \{ x_1x_2\ldots x_k \mid k \geq 0 \text{ and each } x_i \in A \} \]

These are operations on sets of strings!
Closure of … under …

• $\mathbb{Z}$ under addition.
• Set of even ints under multiplication.
• $\{0\}^*$ under concatenation.

Which of these is true?

A. The set of odd integers is closed under addition.
B. The set of positive integers is closed under subtraction.
C. The set of rational numbers is closed under multiplication.
D. The set of real numbers is closed under division.
E. I don't know.
Complementation

Claim:
If $A$ is a regular language, then so is its complement $A$. 

Same as:
If $A = L(M)$ for some DFA $M$, then $A = L(M')$ for some DFA $M'$. 

Proof Strategy: Show that any DFA $M$ can be transformed into a DFA $M'$ such that $L(M') = L(M)$. 
Complementation

**Claim**: If \( A \) is a regular language, then so is \( A \).

**Proof**:

1) Assume \( A \) is regular

2) By definition \( A = L(M) \) for some DFA \( M = (Q, \Sigma, \delta, s, F) \)

3) Let \( M' = (Q, \Sigma, \delta, s, F') \)

4) Claim: \( A = L(M') \)

5) Therefore \( A \) is also regular

How would you define \( F' \)?

A) \( F' = Q - \{s\} \)
B) \( F' = F - \{s\} \)
C) \( F' = Q - F \)
D) \( F' = \{\} \)
Complementation (Proof details)

Claim: Let \( M=(Q, \Sigma, \delta, s, F) \) and \( M'=(Q, \Sigma, \delta, s, \overline{F}) \) be DFAs. Then \( L(M') = \overline{L(M)} \).

Proof:

- \((w \in L(M')) \rightarrow (w \in \overline{L(M)})\)
  1) Assume \( w \) is in \( L(M') \)
  2) By definition of \( L(M') \), \( \delta^*(s, w) \) is in \( F \)
  3) So, \( \delta^*(s, w) \) is not in \( F \), and \( w \) is not in \( L(M) \)
  4) Therefore, \( w \) is in \( \overline{L(M)} \)

- \((w \in \overline{L(M)}) \rightarrow (w \in L(M'))\): similar proof
**Theorem:** The class of regular languages is closed under the union operation.

**Proof:**

What are we proving here?

- A. For any set $A$, if $A$ is regular then so is $A \cup A$.
- B. For any sets $A$ and $B$, if $A \cup B$ is regular, then so is $A$.
- C. For two DFAs $M_1$ and $M_2$, $M_1 \cup M_2$ is regular.
- D. None of the above.
- E. I don't know.
Theorem: The class of regular languages over fixed alphabet Σ is closed under the union operation.

Proof: Let A1, A2 be any two regular languages over Σ. WTS that A1 U A2 is regular.

Goal: build a machine that recognizes A1 U A2.
Goal: build a machine that recognizes $A_1 \cup A_2$.

Strategy: use machines that recognize each of $A_1$, $A_2$.

M1
M2

Input
Accept if either (or both) accepts

** HOW? **
Theorem: The class of regular languages over fixed alphabet $\Sigma$ is closed under the union operation.

Proof: Let $A_1, A_2$ be any two regular languages over $\Sigma$. Given $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ such that $L(M_1) = A_1$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ such that $L(M_2) = A_2$ and \textbf{WTS} that $A_1 \cup A_2$ is regular.

Define $M = (Q_1 \times Q_2, \Sigma, \delta, ?, ?)$

What should be the initial state of $M$?

A. $q_0$
B. $q_1$
C. $q_2$
D. $(q_1, q_2)$
E. I don't know.
Theorem: The class of regular languages over fixed alphabet $\Sigma$ is closed under the union operation.

Proof: Let $A_1$, $A_2$ be any two regular languages over $\Sigma$. Given $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ such that $L(M_1) = A_1$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ such that $L(M_2) = A_2$ and WTS that $A_1 \cup A_2$ is regular.

Define $M = (Q_1 \times Q_2, \Sigma, \delta, ?, ?)$ .

Idea: run in parallel
Theorem: The class of regular languages over fixed alphabet $\Sigma$ is closed under the union operation.

Proof: Let $A_1, A_2$ be any two regular languages over $\Sigma$. Given $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ such that $L(M_1) = A_1$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ such that $L(M_2) = A_2$ and WTS that $A_1 \cup A_2$ is regular. Define $M = (Q_1 \times Q_2, \Sigma, \delta, ?, ?)$

When $r$ is a state in $M_1$, $s$ is a state in $M_2$, and $x$ is in $\Sigma$, then $\delta( (r,s), x ) =$

A. $(r,s)$
B. $(\delta(r,x), \delta(s,x))$
C. $(\delta_1(r,x), s)$
D. $(\delta_1(r,x), \delta_2(s,x))$
E. I don't know.
Theorem: The class of regular languages over fixed alphabet $\Sigma$ is closed under the union operation.

Proof: Let $A_1$, $A_2$ be any two regular languages over $\Sigma$. Given $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ such that $L(M_1) = A_1$ and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ such that $L(M_2) = A_2$ and WTS that $A_1 \cup A_2$ is regular. Define $M = (Q_1 \times Q_2, \Sigma, \delta, ?, ?)$ where $Q = Q_1 \times Q_2$, $\delta$ is the product of $\delta_1$ and $\delta_2$, $q_0 = (q_1, q_2)$, and $F = F_1 \times F_2$.

The set of accepting states for $M$ is

A. $F_1 \times F_2$
B. $\{ (r, s) \mid r \text{ is in } F_1 \text{ and } s \text{ is in } F_2 \}$
C. $\{ (r, s) \mid r \text{ is in } F_1 \text{ or } s \text{ is in } F_2 \}$
D. $F_1 \cup F_2$
E. I don't know.
Union Sipser Theorem 1.25 p. 45

Proof: Let $A_1$, $A_2$ be any two regular languages over $\Sigma$. Given $M_1 = (Q_1,\Sigma,\delta_1,q_1,F_1)$ such that $L(M_1) = A_1$ and $M_2 = (Q_2,\Sigma,\delta_2,q_2,F_2)$ such that $L(M_2) = A_2$. WTS that $A_1 \cup A_2$ is regular. Define $M = (Q_1 \times Q_2,\Sigma,\delta,(q_1,q_2),\{(r,s) \in Q_1 \times Q_2 \mid r \in F_1 \text{ or } s \in F_2\})$ with $\delta((r,s),x) = (\delta_1(r,x),\delta_2(s,x))$ for each $(r,s)$ in $Q_1 \times Q_2$ and $x$ in $\Sigma$. Claim that $L(M) = A_1 \cup A_2$. Proof…
Intersection

• How would you prove that the class of regular languages is closed under intersection?
• Can you think of more than one proof strategy?

\[ A \cap B = \{ x | x \text{ in } A \text{ and } x \text{ in } B \} \]
For next time

Start working on

1) HW2 (discussion: Today, Due: Tuesday)
2) Haskell 1 (discussion: Tuesday, Due: Friday)

Next Time:
Class of regular languages is also closed under concatenation and Kleene star, but harder to prove