Today’s lecture

• Reductions, Reductions, Reductions!
• Did I say reductions?
• More reductions
• Reading: Chapter 5
Reduction from A to B

• A < B, “If B is decidable, then A is decidable”

• Proof:
  – Assume A is decidable
  – Show that B is also decidable

• Proof methods:
  – Using closure properties of decidable languages: transform A into B
  – Let M be a decider for A. Use M to build a decider for B

• Notice: We do not need to know if A is decidable
Getting the direction right

• Different ways to say/write the same thing:
  - “A (Turing) reduces to B”
  - “A < B”
  - If B is decidable, then A is decidable
  - Less common: “B reduces from A”

• A < B: “Hardness of A < Hardness of B”
  - Decidable < Decidable
  - Decidable < Undecidable
  - Undecidable < Undecidable
Applying Reductions

• Reduction from A to B: \( A < B \)

• Proving a reduction:
  – Assume B is decidable
  – Show A is decidable

• Using reductions: \( A < B \)
  – “Hardness of A < Hardness of B”
  – Can be used to show that A is decidable, or B is undecidable
Applying Reductions

- Reduction from A to B: $A < B$
- Proving a reduction:
  - Assume B is decidable
  - Show A is decidable
- Using reductions: $A < B$
  - “Hardness of A < Hardness of B”
  - Can be used to show that A is decidable, or B is undecidable

Goal: Prove that A is **decidable**
What should you do?
A) Show $A < A$
B) Show $B < A$ for some decidable B
C) Show $B < A$ for some undecidable B
D) Show $A < B$ for some decidable B
Applying Reductions

- Reduction from A to B: \( A < B \)
- Proving a reduction:
  - Assume B is decidable
  - Show A is decidable
- Using reductions: \( A < B \)
  - “Hardness of A < Hardness of B”
  - Can be used to show that A is decidable, or B is undecidable

**Goal:** Prove that A is undecidable
What should you do?
A) Show A<A
B) Show B<A for some decidable B
C) Show B<A for some undecidable B
D) Show A<B for some undecidable B
Some example languages

- **Acceptance problem:**
  - $A_{DFA} = \{ <M,w> | M \text{ is a DFA and } M(w) \text{ accepts} \}$
  - $A_{TM} = \{ <M,w> | M \text{ is a TM and } M(w) \text{ accepts} \}$

- **Emptyness problem:**
  - $E_{DFA} = \{ <M> | M \text{ is a DFA and } L(M) \text{ is the empty set} \}$
  - $E_{TM} = \{ <M> | M \text{ is a TM and } L(M) \text{ is the empty set} \}$

- **Equivalence problem:**
  - $EQ_{DFA} = \{ <M,M'> | M \text{ and } M' \text{ are DFAs and } L(M) = L(M') \}$
Some problems on CFG

- $\text{EQ}_{\text{CFG}} = \{ <G_1, G_2> \mid G_1, G_2 \text{ CFG s.t. } L(G_1) = L(G_2) \}$
- $\text{SUB}_{\text{CFG}} = \{ <G_1, G_2> \mid G_1, G_2 \text{ CFG s.t. } L(G_1) \subseteq L(G_2) \}$
- $\text{SUP}_{\text{CFG}} = \{ <G_1, G_2> \mid G_1, G_2 \text{ CFG s.t. } L(G_1) \supseteq L(G_2) \}$

Can you give reductions between any two of these problems? In what direction?
- $\text{EQ}_{\text{CFG}} < \text{SUB}_{\text{CFG}}$?
- $\text{SUB}_{\text{CFG}} < \text{SUP}_{\text{CFG}}$?
- $\text{SUP}_{\text{CFG}} < \text{EQ}_{\text{CFG}}$?
Reduction: $\text{SUB}_{\text{CFG}} < \text{SUP}_{\text{CFG}}$

- $\text{SUB}_{\text{CFG}} = \{ <G_1, G_2> | G_1, G_2 \text{ CFG s.t. } L(G_1) \subseteq L(G_2) \}$
- $\text{SUP}_{\text{CFG}} = \{ <G_1, G_2> | G_1, G_2 \text{ CFG s.t. } L(G_1) \supseteq L(G_2) \}$
- Assume $P$ decides $\text{SUP}_{\text{CFG}}$
- $P'( <G_1, G_2> ) = P( <G_2, G_1> )$

Which statement is false?

A) $P'$ is a decider
B) $P'$ recognizes $\text{SUB}_{\text{CFG}}$
C) $P'$ does not decide $\text{SUP}_{\text{CFG}}$
D) $P'$ decides the complement of $\text{SUP}_{\text{CFG}}$
Reduction: $\text{EQ}_{\text{CFG}} < \text{SUB}_{\text{CFG}}$

- $\text{SUB}_{\text{CFG}} = \{<G_1,G_2> | G_1,G_2 \text{ CFG s.t. } L(G_1) \subseteq L(G_2) \}$
- $\text{EQ}_{\text{CFG}} = \{<G_1,G_2> | G_1,G_2 \text{ CFG s.t. } L(G_1) = L(G_2) \}$
- Assume $P$ decides $\text{SUB}_{\text{CFG}}$
- $P'(G_1,G_2) =$
  1. Run $P(G_1,G_2)$
  2. Run $P(G_2,G_1)$
  3. Accept iff both accepted

Which statement is false?

A) $P'$ is a decider
B) $P'$ recognizes $\text{SUB}_{\text{CFG}}$
C) $P'$ decides $\text{SUB}_{\text{CFG}} \cap \text{SUP}_{\text{CFG}}$
D) $P'$ recognizes $\text{EQ}_{\text{CFG}}$
Reduction: \( \text{SUP}_{\text{CFG}} < \text{EQ}_{\text{CFG}} \)

- \( \text{SUP}_{\text{CFG}} = \{ <G_1,G_2> | G_1,G_2 \text{ CFG s.t. } L(G_2) \subseteq L(G_1) \} \)
- \( \text{EQ}_{\text{CFG}} = \{ <G_1,G_2> | G_1,G_2 \text{ CFG s.t. } L(G_1) = L(G_2) \} \)
- Assume \( P \) decides \( \text{EQ}_{\text{CFG}} \)
- \( P'( <G_1,G_2> ) = \)
  1. Let \( L(G) = L(G_1) \cup L(G_2) \)
  2. Run \( P(<G?,G?>) \)
  3. Accepts iff \( P \) accepts

What input should \( P' \) pass to \( P \) in order to decide \( \text{SUP}_{\text{CFG}} \)?

A) \( <G_1,G_2> \)
B) \( <G_2,G_1> \)
C) \( <G_1,G> \)
D) \( <G_2,G> \)
E) None of the above works
Undecidable Problems

- \( \text{ALL}_{\text{CFG}} = \{ <G> \mid G \text{ is a CFG and } L(G) = \Sigma^* \} \)
- Sipser Theorem 5.13: \( \text{ALL}_{\text{CFG}} \) is undecidable
- What can you say about \( \text{EQ}_{\text{CFG}} \)?
  - \( \text{ALL}_{\text{CFG}} < \text{EQ}_{\text{CFG}} \)
  - Assume \( P \) decides \( \text{EQ}_{\text{CFG}} \)
  - Let \( P'( <G> ) = P( <G, \ "S\rightarrow aS | bS | \ldots | \varepsilon" \ > ) \)
  - \( P' \) decides \( \text{ALL}_{\text{CFG}} \)
  - \( \text{EQ}_{\text{CFG}} \) is undecidable
Undecidable Problems

• \( \text{ALL}_{\text{CFG}} = \{ \langle G \rangle \mid G \text{ is a CFG and } L(G) = \Sigma^* \} \)

• Sipser Theorem 5.13: \( \text{ALL}_{\text{CFG}} \) is undecidable

• What can you say about \( \text{EQ}_{\text{CFG}} \)?
  - \( \text{ALL}_{\text{CFG}} < \text{EQ}_{\text{CFG}} \)
  - Assume \( P \) decides \( \text{EQ}_{\text{CFG}} \)
  - Let \( P'(\langle G \rangle) = P(\langle G, \text{"S→ aS | bS | … | ε"} \rangle) \)
  - \( P' \) decides \( \text{ALL}_{\text{CFG}} \)
  - \( \text{EQ}_{\text{CFG}} \) is undecidable

SUB \( \text{CFG} \) is also undecidable. Which of the following is a valid justification?

A) \( \text{SUB}_{\text{CFG}} < \text{EQ}_{\text{CFG}} \)
B) \( \text{SUB}_{\text{CFG}} < \text{SUP}_{\text{CFG}} \)
C) \( \text{EQ}_{\text{CFG}} < \text{SUB}_{\text{CFG}} \)
D) \( \text{SUB}_{\text{CFG}} < \text{ALL}_{\text{CFG}} \)
$E_{TM}$ is undecidable

- $A_{TM} = \{ <M,w> | M \text{ is a TM and } M(w) \text{ accepts} \}$
- $E_{TM} = \{<M> | M \text{ is a TM and } L(M) \text{ is empty} \}$
- We already proved that $A_{TM}$ is undecidable
- How can we prove that $E_{TM}$ is undecidable?
  - Assume $E_{TM}$ is undecidable
  - Derive a contradiction. E.g., show that $A_{TM}$ is decidable
- Formally: Reduce $A_{TM} < E_{TM}$
$E_{TM}$ is undecidable: Proof

- $A_{TM} = \{ <M,w> \mid M$ is a TM and $M(w)$ accepts $\}$
- $E_{TM} = \{ <M> \mid M$ is a TM and $L(M)$ is empty $\}$

• Assume $E_{TM}$ is decided by $P$

• Define $P'(M,w) =$
  1. Build $M'(x) =$ “if ($x == w$) then $M(x)$ else reject”
  2. Run $P(<M'>)$
  3. If $P$ accepts, then reject. If $P$ rejects, then accept.

• $P'$ decides $A_{TM}$, contradiction! (Sipser Theorem 5.2)
$E_{\text{TM}}$ is undecidable:

- $A_{\text{TM}} = \{ <M,w> \mid M \text{ is a TM and } M(w) \text{ accepts} \}$
- $E_{\text{TM}} = \{<M> \mid M \text{ is a TM and } L(M) \text{ is empty} \}$

• Assume $E_{\text{TM}}$ is decided by $P$

• Define $P'( <M,w> ) =$
  1. Build $M'(x) = \text{"if } (x == w) \text{ then } M(x) \text{ else reject"}$
  2. Run $P(<M'>)$
  3. If $P$ accepts, then reject. If $P$ rejects, then accept.

• $P'$ decides $A_{\text{TM}}$, contradiction!

What can you say about $L(M')$?

A) $L(M') = L(M)$
B) $L(M') = \{x \mid x == w\}$
C) $L(M') \subseteq \{w\}$
D) $w \in L(M')$
E) None of the above
\( \text{E}_{\text{TM}}: \text{Alternative Proof} \)

- Assume \( \text{E}_{\text{TM}} \) is decided by \( P \)
- Let \( P'(\langle M, w \rangle) = \neg(P(\langle M' \rangle)) \)
  
  where \( M'(x) = M(w) \)  
  
  // discard \( x \), and run \( M \) on \( w \)!

- \( P' \) decides \( \text{A}_{\text{TM}} \), contradiction!
  
  ✔ If \( \langle M, w \rangle \) is in \( \text{A}_{\text{TM}} \), then \( L(M') = \Sigma^* \) and \( P(\langle M' \rangle) \) rejects

  ✗ If \( \langle M, w \rangle \) is not in \( \text{A}_{\text{TM}} \), then \( L(M') = \{\} \) and \( P(\langle M' \rangle) \) accepts
Next Time

- Happy Thanksgiving!
- **Reading**: Sipser Chapter 5
- **HW7** due Tue Nov 29
- One more week of classes and the Finals!