Church-Turing Thesis

- Theorem: TM, 2TM, k-TM, NTM, etc. are all equivalent
- Theorem: TM, \(\lambda\)-calculus, java, etc. are all equivalent
- Church-Turing thesis: any “reasonable” model of computation is equivalent to the TM
Language of a TM

- $L(M) = \{w \mid M \text{ accepts } w\}$
- $M$ may reject or loop on strings not in $L(M)$
- A language $X$ is \textit{recognizable} if $X = L(M)$ for some TM $M$
- A TM $M$ is a decider if $M(w)$ halts on every input $w$
- A language $X$ is \textit{decidable} is $X = L(M)$ for some decider $M$
Decidable vs Recognizable

• If A is decidable then A is recognizable
• If A is decidable then $\overline{A}$ is recognizable
  – Equivalently, we may say that A is co-recognizable
• Summary: If A is decidable, then A is both recognizable and co-recognizable
• Question: If A is both recognizable and co-recognizable, can we conclude that A is decidable?
Decidable vs Recognizable

- If A is decidable then A is recognizable.
- If A is decidable then A is recognizable.
  - Equivalently, we may say that A is co-recognizable.
- Summary: If A is decidable, then A is both recognizable and co-recognizable.
- Question: If A is both recognizable and co-recognizable, can be conclude that A is decidable?

A) Yes
B) No
C) It depends on A
D) I don’t know
Recognizers → Decider

• **Theorem:** If A is recognizable and co-recognizable, then A is decidable

• **Proof:**
  - Let M, M’ be TMs such that L(M)=A, L(M’)=A
  - We prove that A is decidable by giving a decider M” such that L(M”)=A
  - M”(w):
    1) Run M(w). If M(w) accepts, then accept
    2) Run M’(w). If M’(w) accepts, then reject
    3) Otherwise (if both reject), then enter an infinite loop
Theorem: If $A$ is recognizable and co-recognizable, then $A$ is decidable.

Proof:

- Let $M, M'$ be TMs such that $L(M) = A$, $L(M') = A$.
- We prove that $A$ is decidable by giving a decider $M''$ such that $L(M'') = A$.
- $M''(w)$:
  1) Run $M(w)$. If $M(w)$ accepts, then accept.
  2) Run $M'(w)$. If $M'(w)$ accepts, then reject.
  3) Otherwise (if both reject), then enter an infinite loop.

Question: What is the language of $M''$?

A) $L(M'') = A$
B) $L(M'') = A$
C) $L(M'') = A \cup A$
D) It depends on the details of $M$ and $M'$.
Theorem: If $A$ is recognizable and co-recognizable, then $A$ is decidable.

Proof:

- Let $M, M'$ be TMs such that $L(M)=A, L(M')=A$.
- We prove that $A$ is decidable by giving a decider $M''$ such that $L(M'')=A$.
- $M''(w)$:
  1) Run $M(w)$. If $M(w)$ accepts, then accept.
  2) Run $M'(w)$. If $M'(w)$ accepts, then reject.
  3) Otherwise (if both reject), then enter an infinite loop.

Question: Is $M''$ a decider?

A) Yes, because it always terminate
B) No, because it may loop in step 1
C) No, because it may loop in step 2
D) No, because it may loop in step 3
E) I don’t know
Recognizers → Decider

• **Theorem:** If $A$ is recognizable and co-recognizable, then $A$ is decidable

• **Proof:** Let $M, M'$ be TMs such that $L(M) = A, L(M') = \overline{A}$, and define
  - $M''(w)$: for $t=1,2,3,...$
    1) Run $M(w)$ for $t$ steps. If $M(w)$ accepts, then accept
    2) Run $M'(w)$ for $t$ steps. If $M'(w)$ accepts, then reject
    3) Otherwise, continue to next $t$
Recognizers → Deciders

- **Theorem**: If A is recognizable and co-recognizable, then A is decidable

- **Proof**: Let $M, M'$ be TMs such that $L(M) = A$, $L(M') = A^c$, and define
  
  - $M''(w)$: for $t = 1, 2, 3, \ldots$
    
    1) Run $M(w)$ for $t$ steps. If $M(w)$ accepts, then accept
    
    2) Run $M'(w)$ for $t$ steps. If $M'(w)$ accepts, then reject
    
    3) Otherwise, continue to next $t$

**Question**: Is $M''$ a decider?

A) Yes, because it always terminate  
B) No, because it may loop in step 1  
C) No, because it may loop in step 2  
D) No, because of infinite loop “for $t = 1, 2, 3 \ldots$”  
E) I don’t know
**Summary**

- **Theorem**: A language $L$ is decidable if and only if it is both recognizable and co-recognizable: $D = RE \cap \text{coRE}$
Closure properties (D)

• Theorem: If A and B are decidable, then AUB is decidable

• Proof:
  – Let M, M’ be deciders such that L(M)=A, L(M’)=B
  – We build a decider for AUB,

    \[ M''(w) = \]
    1) Run M(w). If M(w) accepts, then accept
    2) Run M’(w). If M’(w) accepts, then accept
    3) Otherwise, reject
Theorem: If A and B are decidable, then $A \cup B$ is decidable.

Proof:
- Let $M, M'$ be deciders such that $L(M) = A$ and $L(M') = B$.
- We build a decider for $A \cup B$,

  $M''(w) =$
  
  1) Run $M(w)$. If $M(w)$ accepts, then accept
  2) Run $M'(w)$. If $M'(w)$ accepts, then accept
  3) Otherwise, reject

Question: Is $M''$ a decider?
A) Yes, because it always terminate
B) No, because it may loop in step 1
C) No, because it may loop in step 2
D) I can’t decide, I am not a decider
## Closure properties

- **Theorem:** If A and B are decidable, then \( A \cup B \) is decidable.

- **Proof:**
  - Let \( M, M' \) be deciders such that \( L(M) = A \) and \( L(M') = B \).
  - We build a decider for \( A \cup B \),

\[
M''(w) = \\
1) \text{Run } M(w). \text{ If } M(w) \text{ accepts, then accept} \\
2) \text{Run } M'(w). \text{ If } M'(w) \text{ accepts, then accept} \\
3) \text{Otherwise, reject}
\]

**Question:** What is the language of \( M'' \)?

- A) \( L(M'') = A \)
- B) \( L(M'') = B \)
- C) \( L(M'') = A \cup B \)
- D) \( L(M'') = A - B \)
- E) I don’t know
Other closure properties

• Decidable languages are closed under
  − Union
  − Intersection
  − Set Complement
  − Set Difference
  − ...

• Proof: similar to the proof for union
Closure properties (Recog. lang)

- Theorem: If A and B are recognizable, then AUB is recognizable

- Proof:
  - Let M, M’ be TMs such that L(M)=A, L(M’)=B
  - We build a TM for AUB,
    $\quad M''(w) =$
    1) Run M(w). If M(w) accepts, then accept
    2) Run M’(w). If M’(w) accepts, then accept
    3) Otherwise, reject
Closure properties

• Theorem: If A and B are recognizable, then AUB is recognizable.

• Proof:
  - Let M, M’ be TMs such that L(M)=A, L(M’)=B
  - We build a TM for AUB,

\[
M''(w) = \\
1) \text{Run } M(w). \text{ If } M(w) \text{ accepts, then accept} \\
2) \text{Run } M'(w). \text{ If } M'(w) \text{ accepts, then accept} \\
3) \text{Otherwise, reject}
\]

Question: Is M” a decider?
A) Yes, because it always terminate
B) No, because it may loop in step 1
C) No, because it may loop in step 2
D) I can’t decide, I am not a decider
Closure properties

- Theorem: If $A$ and $B$ are recognizable, then $A \cup B$ is recognizable.

- Proof:
  - Let $M, M'$ be TMs such that $L(M) = A, L(M') = B$.
  - We build a TM for $A \cup B$,
    
    $$M''(w) =$$
    
    1) Run $M(w)$. If $M(w)$ accepts, then accept.
    2) Run $M'(w)$. If $M'(w)$ accepts, then accept.
    3) Otherwise, reject.

Question: What property best describes the language of $M''$?

A) $L(M'') = A \cup B$
B) $A \subseteq L(M'') \subseteq A \cup B$
C) $A \cap B \subseteq L(M'') \subseteq A \cup B$
D) $L(M'') = A - B$
E) None of the above
Closure Properties (RE)

• Can we fix the proof, and show that RE is closed under union?

• Is RE closed under
  – Union?
  – Intersection?
  – Complement?
  – Set Difference?
Closure of RE under union

• Theorem: If $A$ and $B$ are recognizable, then $A \cup B$ is recognizable

• Proof:
  - Let $M, M'$ be TMs such that $L(M)=A, L(M')=B$
  - We build a TM for $A \cup B$, $M''(w) =$

1. For $t=1,2,3,\ldots$
   1) Run $M(w)$ for $t$ steps. If $M(w)$ accepts, then accept
   2) Run $M'(w)$ for $t$ steps. If $M'(w)$ accepts, then accept
   3) Otherwise, continue to next $t$
Closure of RE under union

- Theorem: If A and B are recognizable, then AUB is recognizable

- Proof:
  - Let M, M’ be TMs such that L(M)=A, L(M’)=B
  - We build a TM for AUB, M’’(w) =
    1. For t=1,2,3,…
      1) Run M(w) for t steps. If M(w) accepts, then accept
      2) Run M’(w) for t steps. If M’(w) accepts, then accept
      3) Otherwise, continue to next t

Question: Is M” a decider?
A) Yes, it always terminate
B) No, because it may loop in step 1
C) No, because it may loop in step 2
D) No, because of infinite loop “for t=1,2,3…”
E) I don’t know
For next time:

- Try to prove closure of RE under intersection
- Prove that if RE were closed under complement, then all recognizable languages would also be decidable
- Reading: Sipser Chapter 3, 4.1.
- Haskell 3: due tonight