Relations

A relation over a set $S$ is a set $R \subseteq S \times S$.
- We write $a R b$ for $(a, b) \in R$.

A relation $R$ is:
- reflexive iff $\forall a \in S : a R a$.
- transitive iff $\forall a, b, c \in S : a R b \land b R c \Rightarrow a R c$.
- symmetric iff $\forall a, b \in S : a R b \Rightarrow b R a$.
- anti-symmetric iff $\forall a, b \in S : a R b \Rightarrow a = b$.

Partial orders

An equivalence class is a relation that is:
- reflexive, transitive, symmetric.

A partial order is a relation that is:
- reflexive, transitive, anti-symmetric.

A partially ordered set (a poset) is a pair $(S, \leq)$ of a set $S$ and a partial order $\leq$ over the set.

Examples of posets: $(2^S, \subseteq)$, $(\mathbb{Z}, \leq)$, $(\mathbb{Z}, \text{divides})$.

Lub and glb

Given a poset $(S, \leq)$, and two elements $a, b \in S$, then the:
- least upper bound (lub) is an element $c$ such that $a \leq c, b \leq c$, and $\forall d \in S : (a \leq d \land b \leq d) \Rightarrow c \leq d$.
- greatest lower bound (glb) is an element $c$ such that $c \leq a, c \leq b$, and $\forall d \in S : (d \leq a \land d \leq b) \Rightarrow d \leq c$.
Lub and glb

Given a poset \((S, \leq)\), and two elements \(a \in S\) and \(b \in S\), then the:

- least upper bound (lub) is an element \(c\) such that \(a \leq c, b \leq c,\) and \(\forall d \in S. (a \leq d \land b \leq d) \implies c \leq d\)
- greatest lower bound (glb) is an element \(c\) such that \(c \leq a, c \leq b,\) and \(\forall d \in S. (d \leq a \land d \leq b) \implies d \leq c\)

lub and glb don’t always exists:

Lattices

- A lattice is a tuple \((S, \sqsubseteq, \bot, \top, \sqcup, \sqcap)\) such that:
  - \((S, \sqsubseteq)\) is a poset
  - \(\forall a \in S. \bot \sqsubseteq a\)
  - \(\forall a \in S. a \sqsubseteq \top\)
  - Every two elements from \(S\) have a lub and a glb
  - \(\sqcup\) is the least upper bound operator, called a join
  - \(\sqcap\) is the greatest lower bound operator, called a meet

Examples of lattices

- Powerset lattice

Examples of lattices

- Powerset lattice

Examples of lattices

- Booleans expressions
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