## Binomial Coefficients

<table>
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<tr>
<th>Lecture A</th>
<th>Tiefenbruck</th>
<th>Tu, Th 11am-12:20pm</th>
<th>Center 119</th>
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<tr>
<td>Lecture B</td>
<td>Tiefenbruck</td>
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<td>Lecture C</td>
<td>Minnes</td>
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http://cseweb.ucsd.edu/classes/fa15/cse21-abc/

Nov 5, 2015
How many four-letter strings have one vowel and three consonants?

There are 5 vowels: AEIOU and 21 consonants: BCDFGHJKLMNPQRSTVWXYZ.

A. $5 \times 21^3$
B. $26^4$
C. $5 + 52$
D. None of the above.
How many four-letter strings have one vowel and three consonants?

There are 5 vowels: AEIOU and 21 consonants: BCDFGHJKLMNPQRSTVWXYZ.

<table>
<thead>
<tr>
<th>Template</th>
<th># Matching</th>
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<tbody>
<tr>
<td>VCCC</td>
<td>$5 \times 21 \times 21 \times 21$</td>
</tr>
<tr>
<td>CVCC</td>
<td>$21 \times 5 \times 21 \times 21$</td>
</tr>
<tr>
<td>CCVC</td>
<td>$21 \times 21 \times 5 \times 21$</td>
</tr>
<tr>
<td>CCCV</td>
<td>$21 \times 21 \times 21 \times 5$</td>
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</table>

Total: $4 \times 5 \times 21^3$
If $A = X_1 \cup X_2 \cup \ldots \cup X_n$ and all $X_i$, $X_j$ disjoint and all $X_i$ have same size, then

$$|X_i| = |A| / n$$

More generally:

There are $n/d$ ways to do a task if it can be done using a procedure that can be carried out in $n$ ways, and for every way $w$, $d$ of the $n$ ways give the same result as $w$ did.
Counting with categories

If $A = X_1 \cup X_2 \cup \ldots \cup X_n$ and all $X_i, X_j$ disjoint and all $X_i$ have same size, then

$$|X_i| = |A| / n$$

More generally:

There are $n/d$ ways to do a task if it can be done using a procedure that can be carried out in $n$ ways, and for every way $w$, $d$ of the $n$ ways give the same result as $w$ did.

Rosen p. 394
If $A = X_1 \cup X_2 \cup \ldots \cup X_n$ and all $X_i$, $X_j$ disjoint and all $X_i$ have same size, then

$$|X_i| = \frac{|A|}{n}$$

Or in other words,

If objects are partitioned into categories of equal size, and we want to think of different objects as being the same if they are in the same category, then

$$\text{# categories} = \frac{\text{(# objects)}}{\text{(size of each category)}}$$
An ice cream parlor has \( n \) different flavors available. How many ways are there to order a two-scoop ice cream cone (where you specify which scoop goes on bottom and which on top, and the two flavors must be different)?

A. \( n^2 \)
B. \( n! \)
C. \( n(n-1) \)
D. \( 2n \)
E. None of the above.
An ice cream parlor has \( n \) different flavors available. How can we use our earlier answer to decide the number of cones, if we count two cones as the same if they have the same two flavors (even if they're in opposite order)?

A. Double the previous answer.
B. Divide the previous answer by 2.
C. Square the previous answer.
D. Keep the previous answer.
E. None of the above.
An ice cream parlor has $n$ different flavors available. How can we use our earlier answer to decide the number of cones, if we count two cones as the same if they have the same two flavors (even if they're in opposite order)?

**Objects:**  
**Categories:**  
**Size of each category:**

$$\text{# categories} = \frac{\text{(# objects)}}{(\text{size of each category})}$$
An ice cream parlor has $n$ different flavors available.

How can we use our earlier answer to decide the number of cones, if we count two cones as the same if they have the same two flavors (even if they're in opposite order)?

**Objects**: cones (where order matters)

**Categories**: flavor pairs (regardless of order)

**Size of each category**:

$$\text{# categories} = \frac{\text{# objects}}{\text{size of each category}}$$
An ice cream parlor has $n$ different flavors available. How can we use our earlier answer to decide the number of cones, if we count two cones as the same if they have the same two flavors (even if they're in opposite order)?

**Objects:** cones $n(n-1)$

**Categories:** flavor pairs (regardless of order)

**Size of each category:** 2

\[
\text{# categories} = \frac{(n)(n-1)}{2}
\]

Avoiding double-counting
How many different colored triangles can we create by tying these three pipe cleaners end-to-end?

A. $3!$
B. $2^3$
C. $3^2$
D. 1
E. None of the above.
How many different colored triangles can we create by tying these three pipe cleaners end-to-end?

**Objects**: all different colored triangles  
**Categories**: physical colored triangles (two triangles are the same if they can be rotated and/or flipped to look alike)  
**Size of each category**:

\[
\# \text{ categories} = \frac{\# \text{ objects}}{\text{size of each category}}
\]
How many different colored triangles can we create by tying these three pipe cleaners end-to-end?

**Objects**: all different colored triangles \(3!\)

**Categories**: physical colored triangles (two triangles are the same if they can be rotated and/or flipped to look alike)

**Size of each category**: \((3)(2)\) three possible rotations, two possible flips

\[
\text{# categories} = \frac{\text{# objects}}{\text{size of each category}} = \frac{6}{6} = 1
\]

Challenge: same question with 4 colors (square)
Fixed-density Binary Strings

How many length $n$ binary strings contain $k$ ones?

For example, $n=6$ $k=4$

Which of these strings matches this example?

A. 101101
B. 1100011101
C. 111011
D. 1101
E. None of the above.

Density is number of ones
How many length \( n \) binary strings contain \( k \) ones?

For example, \( n=6 \) \( k=4 \)

Product rule: How many options for the first bit? the second? the third?

Density is number of ones

Rosen p. 413
How many length $n$ binary strings contain $k$ ones?

For example, $n=6 \ k=4$

Tree diagram: *gets very big & is hard to generalize*
How many length \( n \) binary strings contain \( k \) ones?

For example, \( n=6 \ k=4 \)

Another approach: use a different representation i.e. count with categories

Objects:
Categories:
Size of each category:

\[ \text{# categories} = \left( \frac{\text{# objects}}{\text{size of each category}} \right) \]

Density is number of ones
How many length \( n \) binary strings contain \( k \) ones?

For example, \( n=6 \) \( k=4 \)

Another approach: \textbf{use a different representation} i.e. count with categories

\textbf{Objects}: all strings made up of \( 0_1, 0_2, 1_1, 1_2, 1_3, 1_4 \)

\textbf{Categories}: strings that agree except subscripts

\textbf{Size of each category}: Subscripts so objects are distinct

\[ \# \text{ categories} = \frac{\# \text{ objects}}{\text{size of each category}} \]
How many length \( n \) binary strings contain \( k \) ones?

For example, \( n=6 \) \( k=4 \)

Another approach: use a different representation i.e. count with categories

**Objects**: all strings made up of \( 0_1, 0_2, 1_1, 1_2, 1_3, 1_4 \) \( = 6! \)

**Categories**: strings that agree except subscripts

**Size of each category**: ?

\[
\text{# categories} = \frac{\text{(# objects)}}{\text{(size of each category)}}
\]
How many subscripted strings i.e. rearrangements of the symbols
\[ 0_1, 0_2, 1_1, 1_2, 1_3, 1_4 \]
result in
\[ 101101 \]
when the subscripts are removed?

A. \( 6! \)
B. \( 4! \)
C. \( 2! \)
D. \( 2!4! \)
E. None of the above
How many length $n$ binary strings contain $k$ ones?

For example, $n=6$ $k=4$

Another approach: use a different representation i.e. count with categories

**Objects**: all strings made up of $0_1$, $0_2$, $1_1$, $1_2$, $1_3$, $1_4$ $6!$

**Categories**: strings that agree except subscripts

**Size of each category**: $4!2!$

\[
\text{# categories} = \frac{\text{(# objects)}}{\text{(size of each category)}}
\]

\[
= \frac{6!}{(4!2!)}
\]
How many length \( n \) binary strings contain \( k \) ones?

Another approach: use a different representation i.e. count with categories

**Objects**: all strings made up of \( 0_1, 0_2, \ldots, 0_{n-k}, 1_1, 1_2, \ldots, 1_k \)

**Categories**: strings that agree except subscripts

**Size of each category**: \( k!(n-k)! \)

\[
\text{# categories} = \frac{\text{(# objects)}}{\text{(size of each category)}} = \frac{n!}{k!(n-k)!}
\]
A **permutation** of \( r \) elements from a set of \( n \) distinct objects is an **ordered** arrangement of them. There are

\[
P(n,r) = n(n-1)(n-2) \ldots (n-r+1)
\]

many of these.

A **combination** of \( r \) elements from a set of \( n \) distinct objects is an **unordered** selection of them. There are

\[
C(n,r) = \frac{n!}{( r! \ (n-r) ! )} = \binom{n}{r}
\]

many of these.
How many length $n$ binary strings contain $k$ ones?

How to express this using the new terminology?

A. $C(n,k)$
B. $C(n,n-k)$
C. $P(n,k)$
D. $P(n,n-k)$
E. None of the above

Rosen p. 413
How many length $n$ binary strings contain $k$ ones?

How to express this using the new terminology?

A. $C(n,k)$ \{1,2,3..n\} is set of positions in string, choose k positions for 1s
B. $C(n,n-k)$ \{1,2,3..n\} is set of positions in string, choose n-k positions for 0s
C. $P(n,k)$
D. $P(n,n-k)$
E. None of the above
An ice cream parlor has $n$ different flavors available.

How many ice cream cones are there, if we count two cones as the same if they have the same two flavors (even if they're in opposite order)?

**Objects:** cones $n(n-1)$  
**Categories:** flavor pairs (regardless of order)  
**Size of each category:** 2

# categories = $(n)(n-1)/2$

Order doesn't matter so selecting a subset of size 2 of the $n$ possible flavors:

$$C(n,2) = n! / (2! (n-2)!) = n(n-1)/2$$
Binomial: sum of two terms, say $x$ and $y$.

What do powers of binomials look like?

$$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$$
$$= (x^2+2xy+y^2)(x^2+2xy+y^2)$$
$$= (x^4+2x^3y+x^2y^2)+(2x^3y+4x^2y^2+2xy^3)+(x^2y^2+2xy^3+y^4)$$
$$= x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

In general, for $(x+y)^n$

A. All terms in the expansion are (some coefficient times) $x^k y^{n-k}$ for some $k$, $0 \leq k \leq n$.
B. All coefficients in the expansion are integers between 1 and $n$.
C. There is symmetry in the coefficients in the expansion.
D. The coefficients of $x^n$ and $y^n$ are both 1.
E. None of the above.
Binomial Theorem

\[(x+y)^n = (x+y)(x+y)\ldots(x+y)\]

\[= x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{k}x^{n-k}y^k + \ldots + x^2y^{n-2} + xy^{n-1} + y^n\]

Number of ways we can choose \(k\) of the \(n\) factors (to contribute to \(x\)) and hence also \(n-k\) of the factors (to contribute to \(y\))

\[\text{need exactly } k \text{ } x\text{'s which } k \text{ of the factors should contribute an } x?\]
Binomial Theorem

\[(x+y)^n = (x+y)(x+y) \ldots (x+y)\]

\[= x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{k} x^{n-k} y^k + \ldots + \binom{n}{n-1} x y^{n-1} + y^n\]

Number of ways we can choose \(k\) of the \(n\) factors (to contribute to \(x\)) and hence also \(n-k\) of the factors (to contribute to \(y\)) \(\binom{n}{k}\)

\[= x^n + \binom{n}{1} x^{n-1} y + \ldots + \binom{n}{k} x^{n-k} y^k + \ldots + \binom{n}{n-1} x y^{n-1} + y^n\]

\[\binom{n}{n-1} \quad \binom{n}{n-k}\]
What's an identity? An equation that is always true.

To prove $LHS = RHS$

- Use algebraic manipulations of formulas

OR

- Interpret each side as counting some collection of strings, and then prove a statements about those sets of strings
Symmetry Identity

Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Rosen p. 411
Theorem: \[ \binom{n}{k} = \binom{n}{n-k} \]

Proof 1: Use formula
\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}
\]
Symmetry Identity

Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof 1: Use formula
\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}
\]

Proof 2: Combinatorial interpretation?

- **LHS** counts number of binary strings of length \( n \) with \( k \) ones
- **RHS** counts number of binary strings of length \( n \) with \( n-k \) ones

*Rosen p. 411*
Symmetry Identity

Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof 1: Use formula

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}
\]

Proof 2: Combinatorial interpretation?

LHS counts number of binary strings of length n with k ones and n-k zeros

RHS counts number of binary strings of length n with n-k ones and k zeros

\[ 0 \ 0 \ 1 \ 0 \ 0 \ \leftrightarrow \ 1 \ 1 \ 0 \ 1 \ 1 \]
Symmetry Identity

Theorem: \( \binom{n}{k} = \binom{n}{n-k} \)

Proof 1: Use formula
\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}
\]

Proof 2: Combinatorial interpretation?

LHS counts number of binary strings of length \( n \) with \( k \) ones and \( n-k \) zeros
RHS counts number of binary strings of length \( n \) with \( n-k \) ones and \( k \) zeros

Can match up these two sets by pairing each string with another where 0s, 1s are flipped. This bijection means the two sets have the same size. So LHS = RHS.
Theorem: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

Proof 1: Use formula

Proof 2: Combinatorial interpretation?

LHS counts number of binary strings ???

RHS counts number of binary strings ???

length n+1, k ones

Rosen p. 418
Theorem: \[ \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \]

Proof 2: Combinatorial interpretation?

**LHS** counts number of binary strings of length n+1 that have k ones.

**RHS** counts number of binary strings ???

Length n+1 binary strings with k ones
Theorem:
\[
\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}
\]

Proof 2: Combinatorial interpretation?

**LHS** counts number of binary strings of length n+1 that have k ones.  
**RHS** counts number of binary strings ???

Start with 1  
Start with 0

Rosen p. 418
How many length $n+1$ strings start with 1 and have $k$ ones in total?

A. $C(n+1, k+1)$
B. $C(n, k)$
C. $C(n, k+1)$
D. $C(n, k-1)$
E. None of the above.
How many length n+1 strings start with 0 and have k ones in total?

A. $C(n+1, k+1)$
B. $C(n, k)$
C. $C(n, k+1)$
D. $C(n, k-1)$
E. None of the above.

Pascal's Identity

Rosen p. 418
Pascal's Identity

Theorem: \( \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \)

Proof 2: Combinatorial interpretation?

**LHS** counts number of binary strings of length n+1 that have k ones.

**RHS** counts number of binary strings of length n+1 that have k ones, split into two.

Rosen p. 418
Theorem: \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \)

What set does the **LHS** count?

A. Binary strings of length n that have k ones.
B. Binary strings of length n that start with 1.
C. Binary strings of length n that have any number of ones.
D. None of the above.
Theorem: \[\sum_{k=0}^{n} \binom{n}{k} = 2^n\]

Proof: Combinatorial interpretation?

**LHS** counts number of binary strings of length n that have any number of 1s.

By sum rule, we can break up the set of binary strings of length n into disjoint sets based on how many 1s they have, then add their sizes.

**RHS** counts number of binary strings of length n.

This is the same set so **LHS = RHS**.
Reminders

HW 6 due **Friday 11:59pm** via Gradescope.