Today's Topics:

1. Strong vs regular induction
2. Strong induction examples:
   - Divisibility by a prime
   - Recursion sequence: product of fractions

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**1. Strong induction examples**

**DIVISIBILITY BY A PRIME**

- Prove: \( \forall n \geq 1 \; P(n) \)
  - Base case: \( P(1) \)
  - Regular induction: \( P(n) \rightarrow P(n+1) \)

- Strong induction: \( (P(1) \land \ldots \land P(n)) \rightarrow P(n+1) \)
  - Can use more assumptions to prove \( P(n+1) \)
Example for the power of strong induction

- **Theorem:** For all prices \( p \geq 8 \) cents, the price \( p \) can be paid using only 5-cent and 3-cent coins

- **Proof:**
  - **Base case:** \( 8 = 3 + 5 \), \( 9 = 3 + 3 + 3 \), \( 10 = 5 + 5 \)
  - Assume it holds for all prices \( 1 \ldots p-1 \), prove for price \( p \) when \( p \geq 11 \)
  - **Proof:** since \( p-3 \geq 8 \) we can use the inductive hypothesis for \( p-3 \). To get price \( p \) simply add another 3-cent coin.
  - Much easier than standard induction!

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Example for the power of strong induction

- **Proof:**
  - **Base case:** \( 8 = 3 + 5 \), \( 9 = 3 + 3 + 3 \), \( 10 = 5 + 5 \)
  - Assume it holds for all prices \( 1 \ldots p-1 \), prove for price \( p \) when \( p \geq 11 \)
  - **Proof:** since \( p-3 \geq 8 \) we can use the inductive hypothesis for \( p-3 \). To get price \( p \) simply add another 3-cent coin.
  - This doesn’t give us an algorithm to make the change.
  - This gives us the corollary:
    - All prices \( p > 10 \) can be made with at least one three cent coin.

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2. Strong induction examples

**DIVISIBILITY BY A PRIME**

Definitions and properties for this proof

- **Definitions:**
  - \( n \) is **prime** if \( \forall a,b \in \mathbb{N} : n = ab \rightarrow a = 1 \lor b = 1 \)
  - \( n \) is **composite** if \( n = ab \) for some \( 1 < a,b < n \)

- **Prime or Composite exclusivity:**
  - All integers greater than 1 are either prime or composite (exclusive or—can’t be both).

- **Definition of divisible:**
  - \( n \) is divisible by \( d \) iff \( n = dk \) for some integer \( k \).

- **2 is prime** (you may assume this; it also follows from the definition).
Definitions and properties for this proof (cont.)

- Goes without saying at this point:
- The set of Integers is closed under addition and multiplication
- Use algebra as needed

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Thm: For all integers n greater than 1, n is divisible by a prime number.

Proof (by strong mathematical induction):
Basis step: Show the theorem holds for n = ________.
Inductive step:
Assume [or “Suppose”] that
WTS that

So the inductive step holds, completing the proof.

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A. 0
B. 1
C. 2
D. 3
E. Other/none/more than one

So the inductive step holds, completing the proof.

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Thm: For all integers n greater than 1, n is divisible by a prime number.

Proof (by strong mathematical induction):
Basis step: Show the theorem holds for n = ________.
Inductive step:
Assume [or “Suppose”] that
WTS that

A. For some integer n>1, n is divisible by a prime number.
B. For some integer n>1, k is divisible by a prime number, for all integers k where 2≤k≤n.
C. Other/none/more than one

So the inductive step holds, completing the proof.
Thm: For all integers $n$ greater than 1, $n$ is divisible by a prime number.

Proof (by strong mathematical induction):
Basis step: Show the theorem holds for $n = 2$.
Inductive step:
Assume [or “Suppose”] that $n$ is divisible by a prime number. 
WTS that $n+1$ is divisible by a prime number.

So the inductive step holds, completing the proof.

2. Strong induction examples

Definitions and properties for this proof
- Product less than one: 
  \[ \forall a, b \in \mathbb{Q}, a, b < 1 \rightarrow a \cdot b < 1. \]
- Algebra, etc., as usual
Definition of the sequence:
\[ d_1 = \frac{9}{10} \]
\[ d_2 = \frac{10}{11} \]
\[ d_k = d_{k-1}d_{k-2} \text{ for all integers } k \geq 3 \]

Then: For all integers \( n \in \mathbb{Z}, \, 0 < d_n < 1. \)

Proof (by strong mathematical induction):

Basis step: Show the theorem holds for \( n = \) ______.

Inductive step:
Assume [or “Suppose”] that \( 0 < d_n < 1. \)

WTS that \( 0 < d_{n+1} < 1. \)

So the inductive step holds, completing the proof.
Definition of the sequence:
\[ d_1 = \frac{9}{10}, \quad d_2 = \frac{10}{11}, \quad d_k = (d_{k-1})(d_{k-2}) \text{ for all integers } k \geq 3 \]

Then: For all integers \( n \geq 0 \), \( 0 < d_n < 1 \).

**Proof (by strong mathematical induction):**

**Basis step:** Show the theorem holds for \( n = 1, 2 \).

**Inductive step:** Assume [or “Suppose”] that the theorem holds for \( n \geq 2 \). WTS that \( 0 < d_{n+1} < 1 \).

By definition, \( d_{n+1} = d_n d_{n-1} \).
By the inductive hypothesis, \( 0 < d_{n-1} < 1 \) and \( 0 < d_n < 1 \).
Hence, \( 0 < d_{n+1} < 1 \).

So the inductive step holds, completing the proof.

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3. Fibonacci numbers

**Verifying a solution**

Fibonacci numbers

- 1, 1, 2, 3, 5, 8, 13, 21, ...

**Rule:** \( F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-2} + F_{n-1} \).

**Question:** can we derive an expression for the \( n \)-th term?

**YES!** \( F_n = \frac{1 + \sqrt{5}}{2^n} \left( \frac{1 + \sqrt{5}}{2} \right) - \frac{1 - \sqrt{5}}{2^n} \left( \frac{1 - \sqrt{5}}{2} \right) \)

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**Proof by strong induction.**

**Base case:**

A. \( n = 1 \)
B. \( n = 2 \)
C. \( n = 1 \) and \( n = 2 \)
D. \( n = 1 \) and \( n = 2 \) and \( n = 3 \)
E. Other

**Fibonacci numbers**

- Rule: \( F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-2} + F_{n-1} \).
- We will prove an upper bound:
  \[ F_n \leq r^n, \quad r = \frac{1 + \sqrt{5}}{2} \]
- Proof by strong induction.

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Fibonacci numbers

- Rule: $F_1=1$, $F_2=1$, $F_n=F_{n-2}+F_{n-1}$.
- We will prove an upper bound:
  
  $$F_n \leq r^n, \quad r = \frac{1 + \sqrt{5}}{2}$$

- Proof by strong induction.
- Base case:
  
  A. $n=1$
  B. $n=2$
  C. $n=1$ and $n=2$
  D. $n=1$, $n=2$ and $n=3$
  E. Other

Theorem:

- Base cases: $n=1$, $n=2$
- Inductive step: Assume $F_k = r^k$ for $1 \leq k < n$.
- Inductive step: Show...
Fibonacci numbers

- Rule: $F_1=1, F_2=1, F_n=F_{n-2}+F_{n-1}$.
- Theorem: $F_n \leq r^n, \quad r = \frac{1+\sqrt{5}}{2}$
- Base cases: $n=1, n=2$
- Inductive step: show...

Inductive step: need to show $F_n \leq r^n$.

What can we use?
- Definition of $F_n$: $F_n = F_{n-2} + F_{n-1}$
- Inductive hypothesis: $F_{n-1} \leq r^{n-1}, \quad F_{n-2} \leq r^{n-2}$

That is, we need to show that $r^{n-2} + r^{n-1} \leq r^n$

Finishing the inductive step.

- Need to show: $r^{n-2} + r^{n-1} \leq r^n$
- Simplifying, need to show: $1 + r \leq r^2$
- Choice of $r = \frac{1+\sqrt{5}}{2}$ actually satisfied $1 + r = r^2$

(this is why we chose it!)

QED