Today’s Topics:

1. Mathematical Induction Proof
   - 3-cents and 5-cents example
   - Our first algorithm.

3-cent and 5-cent coins

- We will prove the following theorem

Theorem: For all prices \( p \geq 8 \) cents, the price \( p \) can be paid using only 5-cent and 3-cent coins

1851-1889

1866-today
Thm: For all prices $p \geq 8$ cents, the price $p$ can be paid using only 5-cent and 3-cent coins.

Proof (by mathematical induction):
Basis step: Show the theorem holds for price $p=8$.

Inductive step:
Assume [or "Suppose"] that

WTS that

So the inductive step holds, completing the proof.
Thm: For all prices $p \geq 8$ cents, the price $p$ can be paid using only 5-cent and 3-cent coins.

Proof (by mathematical induction):
Basis step: Show the theorem holds for price $p=8$.

Inductive step:
Assume [or "Suppose"] that the theorem is true for some $p \geq 8$.

WTS that

So the inductive step holds, completing the proof.
3-cent and 5-cent coins

- Inductive step:
  - Assume price $p \geq 8$ can be paid using only 3-cent and 5-cent coins.
  - Need to prove that price $p+1$ can be paid using only 3-cent and 5-cent coins.

- Main idea: “reduce” from price $p+1$ to price $p$.

Making change

If we have 100 5-cent coins, and 100 3-cent coins (for a total of $p = \$8.00$), how can we modify the number of 5-cent and 3-cent coins so that we can make the $p+1$ price ($p+1 = \$8.01$)?

A. 40 5-cent coins + 200 3-cent coins
B. 39 5-cent coins + 202 3-cent coins
C. 99 5-cent coins + 102 3-cent coins

Turning our modification scheme into a generic algorithm

If we have $n$ 5-cent coins, and $m$ 3-cent coins (for a total of $p = 5n+3m$), how can we modify the number of 5-cent and 3-cent coins so that we can make the $p+1$ price ($p+1 = 5n+3m+1$)?

A. $n+1$ 5-cent coins + $m-2$ 3-cent coins
B. $n-1$ 5-cent coins + $m+2$ 3-cent coins
C. $n+1$ 5-cent coins + $m+2$ 3-cent coins
D. No generic way
Turning our modification scheme into a generic algorithm

- If we have \( n \) 5-cent coins, and \( m \) 3-cent coins (for a total of \( p = 5n + 3m \)), how can we modify the number of 5-cent and 3-cent coins so that we can make the \( p+1 \) price (\( p+1 = 5n + 3m + 1 \))?
  - A. \( n+1 \) 5-cent coins + \( m-2 \) 3-cent coins
  - B. \( n-1 \) 5-cent coins + \( m+2 \) 3-cent coins
  - C. \( n+1 \) 5-cent coins + \( m+2 \) 3-cent coins
  - D. No generic way

What if we don’t have any 5-cent coins to subtract??

- If we have 0 5-cent coins, and \( m \) 3-cent coins (for a total of \( p = 3m \)), how can we modify the number of 5-cent and 3-cent coins so that we can make the \( p+1 \) price (\( p+1 = 3m + 1 \))?
  - A. You can’t
  - B. You can [explain to your group how]

What if we don’t have any 5-cent coins to subtract??

- If we have 0 5-cent coins, and \( m \) 3-cent coins (for a total of \( p = 3m \)), how can we modify the number of 5-cent and 3-cent coins so that we can make the \( p+1 \) price (\( p+1 = 3m + 1 \))?
  - A. Uh-oh, our proof can not work as we’ve done it so far
  - B. That could never happen [explain why not]
  - C. That could happen, and we need to make a 3rd (or more) case(s) to handle it

That algorithm relies on being able to subtract three 3-cent coins. What if we don’t have that many? (only 1 or 2?)

- A. Uh-oh, our proof can not work as we’ve done it so far
- B. That could never happen [explain why not]
- C. That could happen, and we need to make a 3rd (or more) case(s) to handle it
Thm: For all prices \( p \geq 8 \) cents, the price \( p \) can be paid using only 5-cent and 3-cent coins.

Proof (by mathematical induction):

Basis step: Show the theorem holds for \( p=8 \) (by example, e.g. \( p=3+5 \)).

Inductive step:

Assume [or “Suppose”] that the theorem holds for some \( p \geq 8 \).

WTS that the theorem holds for \( p+1 \).

Assume that \( p=5n+3m \) where \( n,m \geq 0 \) are integers. We need to show that \( p+1=5a+3b \) for integers \( a,b \geq 0 \). Partition to cases:

Case I: \( n \geq 1 \). In this case, \( p+1=5(n-1)+3(m+2) \).

Case II: \( m \geq 3 \). In this case, \( p+1=5(n+2)+3(m-3) \).

Case III: \( n=0 \) and \( m \leq 2 \). Then \( p=5n+3m \leq 6 \) which is a contradiction to \( p \geq 8 \).

So the inductive step holds, completing the proof.

We created an algorithm!

- Our proof actually allows us to algorithmically find a way to pay \( p \) using 3-cent and 5-cent coins.

Algorithm for price \( p \):

1. Start with \( x=8=3+5 \).
2. For \( x=8 \ldots p \), in each step adjust the number of coins according to the modification rules we’ve constructed to maintain price \( x \).
3. Return \( (n,m) \)

Algorithm pseudo-code

PayWithThreeCentsAndFiveCents:

1. Let \( x=8, n=1, m=1 \) (so that \( x=5n+3m \)).
2. While \( x<p \):
   a) \( x:=x+1 \)
   b) If \( n \geq 1 \), set \( n:=n-1, m:=m+2 \)
   c) Otherwise, set \( n:=n+2, m:=m-3 \)
3. Return \( (n,m) \)

We proved that \( n,m \geq 0 \) in this process always; this is not immediate from the algorithm code.
Algorithm run example

- 8 = 0
- 9 = 0
- 10 = 0
- 11 = 0
- 12 = 0

Algorithm properties

- Theorem: Algorithm uses at most two nickels (i.e. \( n \leq 2 \))
- Proof: by induction on \( p \)
  - Try to prove it yourself first!

Proof by cases:
- Case I: \( n \neq 1 \). So \( p + 1 = 5(n - 1) + 3(m + 2) \) and \( a = n - 1 \leq 2 \).
- Case II: \( n = 0 \). So \( p + 1 = 5(2) + 3(m - 3) \). \( a = 2 \).

In both cases \( p + 1 = 5a + 3b \) where \( a \leq 2 \). QED