Today's Topics:
1. Relations
2. Equivalence relations
3. Modular arithmetics

Relations are graphs
- Think of relations as directed graphs
- xRy means “there in an edge x→y”

1. Relations

Options:
A. □ [red] R [yellow] ?
B. □ [yellow] R [red] ?
C. Both
D. Neither
Relations are graphs

What does this relation captures?
xRy means

A. x>y
B. x=y
C. x divides y
D. x+y
E. None/more than one

Types of relations

A relation is symmetric if xRy⇒yRx.
That is, if the graph is undirected

Which of the following is symmetric

A. x<y
B. x divides y
C. x and y have the same sign
D. x≠y
E. None/more than one

Types of relations

A relation is reflexive if xRx is true for all x
That is, the graph has loops in all vertices

Which of the following is reflexive

A. x<y
B. x divides y
C. x and y have the same sign
D. x≠y
E. None/more than one

Types of relations

A relation is transitive if xRy ∧ yRz ⇒ xRz
This is less intuitive… will show equivalent criteria soon

Which of the following is transitive

A. x<y
B. x divides y
C. x and y have the same sign
D. x≠y
E. None/more than one
Types of relations

- A relation is **transitive** if \( xRy \land yRz \Rightarrow xRz \)

- Theorem: Let \( G \) be the graph corresponding to a relation \( R \). \( R \) is transitive iff whenever you can reach from \( x \) to \( y \) in \( G \) then the edge \( x\rightarrow y \) is in \( G \).

  - **Try to prove yourself first**

---

Types of relations

- Theorem: \( R \) is transitive iff when you can reach from \( x \) to \( y \) in \( G \) then the edge \( x\rightarrow y \) is also in \( G \).

  - **Proof (sufficient):**
    - Assume the graph \( G \) has this property. We will show \( R \) is transitive.
    - Let \( x,y,z \) be such that \( xRy \) and \( yRz \) hold.
    - In the graph \( G \) we can reach from \( x \) to \( z \) via the path \( x\rightarrow y\rightarrow z \). So by assumption on \( G \), \( x\rightarrow z \) is also an edge in \( G \).
    - Hence \( xRz \) so \( R \) is transitive.

  - **Proof (necessary) by contradiction:**
    - Assume by contradiction \( R \) is transitive but \( G \) doesn’t have this property.
    - So, there are vertices \( x,y \) with a path \( x\rightarrow v_1\rightarrow \ldots \rightarrow v_k\rightarrow y \) in \( G \) but where the edge \( x\rightarrow y \) is NOT in \( G \).
    - Choose such a pair \( x,y \) with **minimal path length** \( k \).
    - We divide the proof to cases.

  - **Case 1:** \( k=0 \). So \( x\rightarrow y \) in \( G \). Contradiction.
  - **Case 2:** \( k=1 \). Since \( R \) is transitive then \( x\rightarrow v_1 \) and \( v_1\rightarrow y \) imply \( x\rightarrow y \).
    - **Contradiction.**
  - **Case 3:** \( k>1 \). Then \( x\rightarrow v_k \) must be in \( G \) since the path \( x\rightarrow v_1 \rightarrow \ldots \rightarrow v_k \) has length \( k-1 \) and we assumed the path from \( x \) to \( y \) is of minimal length. So in fact \( x\rightarrow v_k\rightarrow y \). Contradiction.

  QED
2. Equivalence relations

A Set Partition $T$ of a set $S$ is a subset of the power set $\mathcal{P}(S)$ such that each set in $T$ is disjoint from any other set in $T$ and the union of all sets in $T$ is $S$.

$T \subset \mathcal{P}(S)$:
- If $A \in T$ and $B \in T$ with $A \neq B$ then $A \cap B = \emptyset$
- If $T = \{A_1, A_2, \ldots, A_k\}$ then $A_1 \cup A_2 \cup \ldots \cup A_k = S$

Example: $\{\{1, 5, 6\}, \{2, 3\}, \{4, 7, 8\}\}$ is a set partition of $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

Equivalence relations
- Definition: a relation is an equivalence relation if it is
  - Reflexive: $\forall x. xRx$
  - Symmetric: $\forall x, y. xRy \leftrightarrow yRx$
  - Transitive: $\forall x, y, z. xRy \land yRz \rightarrow xRz$
- What does that actually mean??
Equivalence relations

- An equivalence relation partitions the universe to equivalence classes
- The set of equivalence classes form a set partition!
- E.g. all people who were born on 11/1/11 is one equivalence class
- Reflexive: a person has the same birthday as himself...
- Symmetric: if x,y have the same birthday then so do y,x
- Transitive: if x,y have the same birthday, and y,z have the same birthday, then so do x,z

Equivalence relations

- As a graph

Which of the following is an equivalence relation in the universe of integer numbers

- A. x divides y
- B. x*y>0
- C. x+y>0
- D. x+y is even
- E. None/more than one/other
Equivalence relations

- Which of the following is an equivalence relation in the universe of graphs

A. x, y have the same number of vertices
B. x, y have the same edges
C. x, y are both Eulerian
D. x, y are the same up to re-labeling the vertices (isomorphic)
E. None/more than one/other

Equivalence relations as functions

- We can see an equivalence relation as a function

  Universe → Property

  E.g. People → birthday
       Integers → sign
       Graphs → #vertices

- An equivalence class is the set of elements mapped to the same value

Modular arithmetics

- We will later see a very important example of an equivalence relation – modular arithmetics

- It has many applications in algorithms and cryptography