1. Functions

Definition of a function

- A function $f: X \rightarrow Y$ is a mapping that maps each element of $X$ to an element of $Y$

- Each element of $X$ is mapped to exactly one element of $Y$
  - Not two
  - Not none
  - Exactly one!
What is a function?

Is the following a function from X to Y?

A. Yes
B. No

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Is the following a function from X to Y?
What is a function?

- Is the following a function from Y to X?
  - A. Yes
  - B. No

Properties of functions

Injective, surjective, bijective

Injective, Surjective, Bijective...

- Function \( f:X \to Y \)
- \( f \) is injective (1 to 1) if: \( f(x) = f(y) \Rightarrow x = y \)
  - That is, no two elements in \( X \) are mapped to the same value
- \( f \) is surjective (onto) if: \( \forall y \in Y \exists x \in X \) s.t. \( f(x) = y \)
  - There is always an "pre-image"
  - Could be more than one \( x \)!
- \( f \) is bijective if it is both injective and surjective
Injective, Surjective, Bijective...

Is the following function

A. Injective
B. Surjective
C. Bijective
D. None

Which of the following functions \( f: \mathbb{N} \rightarrow \mathbb{N} \) is not injective

A. \( f(x) = x \)
B. \( f(x) = x^2 \)
C. \( f(x) = x + 1 \)
D. \( f(x) = 2x \)
E. None/other/more than one
### Injective, Surjective, Bijective...:

#### Which of the following functions $f: \mathbb{R} \rightarrow \mathbb{R}$ is not injective

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#### Which of the following functions $f: \mathbb{N} \rightarrow \mathbb{N}$ is not surjective

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### Inverses

### Injective, Surjective, Bijective...:

#### Which of the following functions $f: \mathbb{R} \rightarrow \mathbb{R}$ is not surjective

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Inverse functions

- Functions $f: X \to Y$ and $g: Y \to X$ are inverses if
  - $\forall x \in X, g(f(x)) = x$
  - $\forall y \in Y, f(g(y)) = y$
- In this case we write $g = f^{-1}$ (and also $f = g^{-1}$)

Inverse functions

- Does the following function have an inverse:
  - $f: \mathbb{R} \to \mathbb{R}, f(x) = 2x$
  - A. Yes
  - B. No

Inverse functions

- Does the following function have an inverse:
  - $f: \mathbb{Z} \to \mathbb{Z}, f(x) = 2x$
  - A. Yes
  - B. No

Inverse functions

- Does the following function have an inverse:
  - $f: \{1,2\} \to \{1,2,3,4\}, f(x) = 2x$
  - A. Yes
  - B. No
Functions with an inverse are surjective

Let $f:X \to Y$, $g:Y \to X$ be inverse functions

**Theorem:** $f$ is surjective

**Proof (by contradiction):**
- Assume not. That is, there is $y \in Y$ such that for any $x \in X$, $f(x) \neq y$.
- Let $x' = g(y)$. Then, $x' \in X$ and $f(x') = y$.
- Contradiction. Hence, $f$ is surjective. QED

Functions with an inverse are injective

Let $f:X \to Y$, $g:Y \to X$ be inverse functions

**Theorem:** $f$ is injective

**Proof (by contradiction):**
- Assume not. That is, there are distinct $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$.
- Then $g(f(x_1)) = g(f(x_2))$.
- But since $f, g$ are inverses, $g(f(x_1)) = x_1$ and $g(f(x_2)) = x_2$.
- So $x_1 = x_2$.
- Contradiction. Hence, $f$ is injective. QED

Functions with an inverse are bijective

Let $f:X \to Y$, $g:Y \to X$ be inverse functions

We just showed that $f$ must be both surjective and injective

Hence, **bijective**

It turns out that the opposite is also true – any bijective function has an inverse. We might prove it later.