Today's Topics:

1. A second look at contradictions
2. Proof by contradiction template
3. Practice negating theorems

P AND \neg P = \text{Contradiction}

- It is not possible for both P and NOT P to be true
- This simply should not happen!
Contradictions destroy the entire system that contains them!

- Draw the truth table:

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>(p &amp; \neg p)</th>
<th>(p &amp; \neg p) \rightarrow q</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

A. 0, 0, 0, 0
B. 1, 1, 1, 1
C. 0, 1, 0, 1
D. 0, 0, 1, 1
E. None/more/other

Here’s the scary part: it doesn’t matter what q is!
- q=“All irrational numbers are integers.”
- q=“The Beatles were a terrible band.”
- q=“Dividing by zero is totally fine!”
- All these can be “proved” true in a system that contains a contradiction!

Proof by contradiction template

- Thm. [write theorem]
- Proof (by contradiction):
  Assume not. That is, suppose that \( \neg \text{[theorem]} \) (Don’t just write \( \neg \text{(theorem)} \). For example, \( \forall \) changes to \( \exists \), use DeMorgan’s law if needed, etc.)
  [write something that leads to a contradiction:
  “… \[p\] … so \( \neg \neg p \). But \[p\], a contradiction!”]
  Conclusion: So the assumed assumption is false and the theorem is true. QED.

Example 1

- Thm. There is no integer that is both even and odd.
- Proof (by contradiction)
  Assume not. That is, suppose
  A. All integers are both odd and even
  B. All integers are not even or not odd.
  C. There is an integer n that is both odd and even.
  D. There is an integer n that is neither even nor odd.
  E. Other/none/more than one
Be careful about doing negations

- Theorem: “there is no integer that is both even and odd”
  - $\neg \exists n \in \mathbb{Z} \ (n \in E \land n \in O)$
  - $\forall n \in \mathbb{Z} \ \neg (n \in E \land n \in O)$

- Negation: $\exists n \in \mathbb{Z} \ (n \in E \land n \in O)$
  - “There is an integer $n$ that is both even and odd”

Example 1

- Thm. There is no integer that is both even and odd.
- Proof (by contradiction)
  Assume not. That is, suppose there exists an integer $n$ that is both even and odd.

Try by yourself first

Conclusion: The assumed assumption is false and the theorem is true. QED.

Example 2

- A number $x$ is rational if $x=a/b$ for integers $a, b$.
  - E.g. $3=3/1, 1/2, -3/4, 0=0/1$

- A number is irrational if it is not rational
  - E.g. $\sqrt{2}$ (proved in textbook)

- Theorem: If $x^2$ is irrational then $x$ is irrational.
Example 2

- Theorem: If \( x^2 \) is irrational then \( x \) is irrational.
- Proof: by contradiction.
  Assume that

  A. There exists \( x \) where both \( x, x^2 \) are rational
  B. There exists \( x \) where both \( x, x^2 \) are irrational
  C. There exists \( x \) where \( x \) is rational and \( x^2 \) irrational
  D. There exists \( x \) where \( x \) is irrational and \( x^2 \) rational
  E. None/other/more than one

Since \( x \) is rational \( x = \frac{a}{b} \) where \( a, b \) are integers.
But then \( x^2 = \frac{a^2}{b^2} \). Both \( a^2, b^2 \) are also integers and hence \( x^2 \) is rational.
A contradiction.

Example 3

- Theorem: \( \sqrt{2} + \sqrt{3} \) is irrational
- Proof (by contradiction).

**Try by yourself first**

**Try by yourself first in groups.**
Example 3
- Theorem: $\sqrt{2} + \sqrt{3}$ is irrational
- Proof (by contradiction).
- Assume not. Then there exist integers $a,b$ such that $\frac{a}{b} = \sqrt{2} + \sqrt{3}$

Squaring gives $a^2 / b^2 = (\sqrt{2} + \sqrt{3})^2 = 2 + 2\sqrt{6} + 3$

So also $\sqrt{6}$ is rational since $\sqrt{6} = (a^2 - 5b^2) / 2b^2$.

[So, to finish the proof it is sufficient to show that $\sqrt{6}$ is irrational.]