Photometric stereo

• Single viewpoint, multiple images under different lighting.
  1. Arbitrary known BRDF, known lighting
  2. Lambertian BRDF, known lighting
  3. Lambertian BRDF, unknown lighting.

Three Source Photometric stereo:
Step 1
Offline:
Using source directions & BRDF, construct reflectance map for each light source direction. \( R_1(p,q), R_2(p,q), R_3(p,q) \)
Online:
1. Acquire three images with known light source directions. \( E_1(x,y), E_2(x,y), E_3(x,y) \)
2. For each pixel location \((x,y)\), find \((p,q)\) as the intersection of the three curves
   \( R_1(p,q) = E_1(x,y) \)
   \( R_2(p,q) = E_2(x,y) \)
   \( R_3(p,q) = E_3(x,y) \)
3. This is the surface normal at pixel \((x,y)\). Over image, the normal field is estimated.

Reflectance Map of Lambertian Surface

What does the intensity (irradiance) of one pixel in one image tell us?
It constrains the surface normal projecting to that point to a curve

Two Light Sources
Two reflectance maps

A third image would disambiguates match
Lambertian Surface

At image location \((u,v)\), the intensity of a pixel \(x(u,v)\) is:

\[
e(u,v) = [a(u,v)\hat{n}(u,v)] \cdot [s_0\hat{s}]
\]

where
- \(a(u,v)\) is the albedo of the surface projecting to \((u,v)\).
- \(\hat{n}(u,v)\) is the direction of the surface normal.
- \(s_0\) is the light source intensity.
- \(\hat{s}\) is the direction to the light source.

Lambertian Photometric stereo

- If the light sources \(s_1, s_2,\) and \(s_3\) are known, then we can recover \(b\) at each pixel from as few as three images. (Photometric Stereo: Silver 80, Woodham 81).

\[
[e_1\ e_2\ e_3] = b^T [s_1\ s_2\ s_3]
\]

- i.e., we measure \(e_1, e_2,\) and \(e_3\) and we know \(s_1, s_2,\) and \(s_3\). We can then solve for \(b\) by solving a linear system.

\[
b^T = [e_1\ e_2\ e_3][s_1\ s_2\ s_3]^{-1}
\]

- Surface normal is: \(n = b/|b|\), albedo is: \(|b|\)

The Space of Images

- Consider an \(n\)-pixel image to be a point in an \(n\)-dimensional space, \(x \in \mathbb{R}^n\).
- Each pixel value is a coordinate of \(x\).
- Many results will apply to linear transformations of image space (e.g., filtered images).
- Other image representations (e.g., Cayley-Klein spaces)

Assumptions

For discussion, we assume:
- Lambertian reflectance functions.
- Objects have convex shape.
- Light sources at infinity.
- Orthographic projection.

- Note: many of these can be relaxed....
Lambertian Assumption with shadowing:

\[ x = \max(Bs, 0) \]

where

- \( x \) is an \( n \)-pixel image vector
- \( B \) is a matrix whose rows are unit normals scaled by the albedos
- \( s \in \mathbb{R}^3 \) is vector of the light source direction scaled by intensity

**Model for Image Formation**

**3-D Linear subspace**

The set of images of a Lambertian surface with no shadowing is a subset of 3-D linear subspace.

\[ L = \{ x \mid x = Bs, \forall s \in \mathbb{R}^3 \} \]

where \( B \) is an \( n \) by 3 matrix whose rows are a product of the surface normal and Lambertian albedo

**Still Life**

Original Images

Basis Images

**Rendering Images**

**How do you construct subspace?**

- Any three images w/o shadows taken under different lighting span \( L \)
- Not orthogonal
- Orthogonalize with Grahm-Schmidt

With more than three images, perform least squares estimation of \( B \) using Singular Value Decomposition (SVD)
Matrix Decompositions

• Definition: The factorization of a matrix $M$ into two or more matrices $M_1, M_2, \ldots, M_n$, such that $M = M_1M_2\ldots M_n$.

• Many decompositions exist...
  - QR Decomposition
  - LU Decomposition
  - LDU Decomposition
  - Etc.

Singular Value Decomposition

• Any $m \times n$ matrix $A$ may be factored such that
  $A = U \Sigma V^T$

• $U$: $m \times m$, orthogonal matrix
  - Columns of $U$ are the eigenvectors of $AA^T$

• $V$: $n \times n$, orthogonal matrix,
  - Columns are the eigenvectors of $A^TA$

• $\Sigma$: $m \times n$, diagonal with non-negative entries ($\sigma_1, \sigma_2, \ldots, \sigma_s$) with $s = \min(m,n)$ are called the singular values
  - Singular values are the square roots of eigenvalues of both $AA^T$ and $A^TA$

• Result of SVD algorithm: $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_s$

SVD Properties

• In Matlab, $[U S V] = \text{svd}(A)$, and you can verify that $A = U\Sigma V^T$

• $r = \text{Rank}(A) = \#$ of non-zero singular values

• $U, V$ give orthonormal bases for the subspaces of $A$:
  - 1st $r$ columns of $U$: Column space of $A$
  - Last $m-r$ columns of $U$: Left nullspace of $A$
  - 1st $r$ columns of $V$: Row space of $A$
  - Last $n-r$ columns of $V$: Nullspace of $A$

• For $d \leq r$, the first $d$ columns of $U$ provide the best $d$-dimensional basis for columns of $A$ in least squares sense.

Thin SVD

• Any $m \times n$ matrix $A$ may be factored such that
  $A = U\Sigma^r V^T$

• If $m > n$, then one can view $\Sigma^r$ as:
  - Where $\Sigma^r = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$ with $r = \min(m,n)$, and lower matrix is $(n-m \times m)$ of zeros.

• Alternatively, you can write:
  $A = U\Sigma^r V^T$

• In Matlab, thin SVD is $[U S V] = \text{svds}(A)$

Estimating $B$ with SVD

1. Construct data matrix $D = [x_1 \ x_2 \ x_3 \ldots \ x_n]$

2. $[U S V] = \text{svds}(D)$

• If data had no noise, then $\text{rank}(D) = 3$, and the first three singular values ($S$) would be positive and rest would be zero.
• Take first three columns of $U$ as $B$.

Face Basis

Original Images

Basis Images
The set of images of a Lambertian surface with no shadowing is a subset of 3-D linear subspace.

\[ L = \{ x | x = Bs, \forall s \in \mathbb{R}^3 \} \]

where \( B \) is a \( n \times 3 \) matrix whose rows are the product of the surface normal and Lambertian albedo \( L_0 \).

The image \( x \) produced by multiple light sources is

\[ x = \sum_{i=0}^{M} \max(Bs_i, 0) \]

• \( x \) is an \( n \)-pixel image vector.
• \( B \) is a matrix whose rows are unit normals scaled by the albedo.
• \( s_i \) is the direction and strength of the light source \( i \).

The set of images of any object in fixed posed, but under all lighting conditions, is a convex cone in the image space.

Theorem: The set of images of any object in fixed posed, but under all lighting conditions, is a convex cone in the image space.

(Belhumeur and Kriegman, IJCV, 98)
Do Ambiguities Exist? Yes

- Cone is determined by linear subspace $L$
- The columns of $B$ span $L$
- For any $A \in \text{GL}(3)$, $B^* = BA$ also spans $L$.
- For any image of $B$ produced with light source $S$, the same image can be produced by lighting $B^*$ with $S^* = A^{-1}S$ because $X = B^*S^* = BAA^{-1}S = BS$
- When we estimate $B$ using SVD, the rows are NOT generally normal * albedo.

Surface Integrability

In general, $B^*$ does not have a corresponding surface.
Linear transformations of the surface normals in general do not produce an integrable normal field.

Generalized Bas-Relief Transformations

Objects differing by a GBR have the same illumination cone.
Without knowledge of light source location, one can only recover surfaces up to GBR transformations.

Uncalibrated photometric stereo

1. Take $n$ images as input, perform SVD to compute $B^*$.
2. Find some $A$ such that $B^*A$ is close to integrable.
3. Integrate resulting gradient field to obtain height function $f(x,y)$.

Comments:
- $f^*(x,y)$ differs from $f(x,y)$ by a GBR.
- Can use specularities to resolve GBR for non-Lambertian surface.
What about cast shadows for nonconvex objects?

P. P. Reubens in Opticorum Libri Sex, 1613

GBR Preserves Shadows

Given a surface \( f \) and a GBR transformed surface \( f' \), then for every light source \( s \) which illuminates \( f \) there exists a light source \( s' \) which illuminates \( f' \) such that the attached and cast shadows are identical.

GBR is the only transform that preserves shadows.

[Kriegman, Belhumeur 2001]

Bas-Relief Sculpture

Codex Urbinas

As far as light and shade are concerned low relief fails both as sculpture and as painting, because the shadows correspond to the low nature of the relief, as for example in the shadows of foreshortened objects, which will not exhibit the depth of those in painting or in sculpture in the round.

Leonardo da Vinci
Treatise on Painting (Kemp)

Some natural ideas & questions

- Can the cones of two different objects intersect?
- Can two different objects have the same cone?
- How big is the cone?
- How can cone be used for recognition?
Two Different Objects

• When a convex Lambertian surface is illumination by perfectly diffuse lighting, the resulting image is directly proportional to the albedo.

• For a convex object, the n-dimensional vector of albedos (and image) is contained within the object’s cone.

• For two objects with the same albedo pattern but different shape, their cones intersect in the interior.

• Two objects differing by a generalized bas relief transformation have the same cone.

Some natural ideas & questions

• Can the cones of two different objects intersect?
• Can two different objects have the same cone?
• How big is the cone?
• How can cone be used for recognition?

The Illumination Cone

Thm: The span of the extreme rays of the illumination cone is equal to the number of distinct surface normals – i.e., as high as n.

The number of extreme rays of the cone is n(n-1)+2

(Bellmaner and Kriegman, IJCV,'98)

Illumination Cone

Shape of the Illumination Cone

Observation: The illumination cone is flat with most of its volume concentrated near a low-dimensional linear subspace.

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<td>96.3</td>
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</tr>
</tbody>
</table>

Dimension: 5 ± 2D

Subsequent results

• Illumination cone is well capture by nine dimensions for a convex Lambertian surface.
  - Spherical Harmonic representation of lighting & BRDF.

Some natural ideas & questions

• Can the cones of two different objects intersect?
• Can two different objects have the same cone?
• How big is the cone?
• How can cone be used for recognition?
Illumination Cones: Recognition Method

- Distance to cone
- Cost O(ne^2) where
  - n: # pixels
  - e: # extreme rays
- Distance to subspace

Generating the Illumination Cone

- Original (Training) Images
- 3D linear subspace
  - $f_0(x, y)$ (albedo) mapped on surface

Predicting Lighting Variation

- Single Light Source

Yale Face Database B

- 64 Lighting Conditions
- 9 Poses
- $\geq 576$ Images per Person

Face Recognition: Test Subsets

- Subset 1: 0-12°
- Subset 2: 12-25°
- Subset 3: 25-50°
- Subset 4: 50-77°

Test images divided into 4 subsets depending on illumination.

Geodesic Dome Database - Frontal Pose

- Error Rate
- Correlation
- Eigenfaces
- 3-D Linear Subspace
- Illumination Cones

[Georghiades, Belhumeur, Kriegman 01]
Face Recognition: Lighting & Pose

1. Union of linear subspaces
   1. Sample the pose space, and for each pose construct illumination cone.
   2. Cone can be approximated by linear subspace (used 11-D)
   2. For computational efficiency, project using PCA to 100-D

Illumination Variability Reveals Shape

Pose Variability in test set: Up to 24°

Illumination Cone Face Recognition Result: Pose and Lighting

Error Rate: 1.02%
Error Rate: 3.4%

Illumination & Image Set

- Lack of illumination invariants [Chen, Jacobs, Belhumeur 98]
- Set of images of Lambertian surface w/o shadowing is 3-D linear subspace [Moses 93], [Nayar, Murase 96], [Shashua 97]
- Empirical evidence that set of images of object is well-approximated by a low-dimensional linear subspace [Hallinan 94], [Epstein, Hallinan, Yuille 95]
- Illumination cones [Belhumeur, Kriegman 98]
- Spherical harmonics lighting & images [Basri, Jacobs 01], [Ramamoorthi, Hanrahan 01]
- Analytic PCA of image over lighting [Ramamoorthi 02]
Some subsequent work

1. “Face Recognition Under Variable Lighting using Harmonic Image Exemplars,” Zhang, Samaras, CVPR03
2. “Clustering Appearances of Objects Under Varying Illumination Conditions,” Ho, Lee, Lim, Kriegman, CVPR 03
3. “Low-Dimensional Representations of Shaded Surfaces under Varying Illumination,” Nillius, Eklundh, CVPR03
4. “Using Specularities for Recognition,” Osadchy, Jacobs, Ramamoorthi, ICCV 03