MATH CSE20 Homework 8 - Solutions

Due Monday November 25

Assigned reading: SF Sections 1 and EO Section 2

(1) (SF 1.15) Which of the following are partitions of $\{1, 2, \ldots, 8\}$? Explain your answers.

(a) $\{\{1, 3, 5\}, \{1, 2, 6\}, \{4, 7, 8\}\}$

Not a partition because 1 appears in two blocks so the blocks aren't disjoint.

(b) $\{\{1, 3, 5\}, \{2, 6, 7\}, \{4, 8\}\}$

Is a partition because the blocks are disjoint (have empty intersection) and each element of the set is in some block.

(c) $\{\{1, 3, 5\}, \{2, 6\}, \{2, 6\}, \{4, 7, 8\}\}$

Note that this set can equivalently be written as $\{\{1, 3, 5\}, \{2, 6\}, \{4, 7, 8\}\}$ and now we see that each element of the original set appears in exactly one block. Thus, this is a partition.

(d) $\{\{1, 5\}, \{2, 6\}, \{4, 8\}\}$.

Not a partition because 3 and 7 are in no blocks.

(2) (SF 1.20) Compare the following pairs of sets. Can they be equal? Is one a subset of the other? Can they have the same size (number of elements)?

(a) $\mathcal{P}(A \cup B)$ and $\mathcal{P}(A) \cup \mathcal{P}(B)$

These sets are equal if $A \subseteq B$ or $B \subseteq A$. If this condition holds, then the two equal sets have the same size. To see that the subset condition guarantees equality, note that if $A \subseteq B$ then $A \cup B = B$ and $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ so $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(B)$. Similar arguments work in the case $B \subseteq A$.

For general A, B, $\mathcal{P}(A \cup B) \supseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ but we may not always have equality.

- Let A, B be arbitrary. Then if $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$ then either $X \subseteq A$ or $X \subseteq B$. In either case, $X \subseteq A \cup B$ so $X \in \mathcal{P}(A \cup B)$.
- To prove that the inclusion may be strict, consider the example $A = \{1\}$ and $B = \{2\}$. Then $\mathcal{P}(A) = \{\varnothing, A\}, \mathcal{P}(B) = \{\varnothing, B\}$ so $\mathcal{P}(A) \cup \mathcal{P}(B) = \{\varnothing, \{1\}, \{2\}\}$. But, $\mathcal{P}(A \cup B) = \mathcal{P}(\{1, 2\}) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\} \supset \{\varnothing, \{1\}, \{2\}\}$.

(b) $\mathcal{P}(A \cap B)$ and $\mathcal{P}(A) \cap \mathcal{P}(B)$

These sets are equal (so, in particular, have the same size). To prove this, consider arbitrary sets A, B. We need to prove subset inclusion in two directions.

- To prove that $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$, let $X \in \mathcal{P}(A \cap B)$ so $X \subseteq A \cap B$. By definition of intersection, this means that $X \subseteq A$ and $X \subseteq B$. Thus, $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$. Hence, $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$, as required.
- Conversely, to prove that $\mathcal{P}(A \cap B) \supseteq \mathcal{P}(A) \cap \mathcal{P}(B)$, let $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Then $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$. Thus, $X \subseteq A$ and $X \subseteq B$. In other words, every element of X is in both A and B. This is

the definition of membership in the intersection, so we have that $X \subseteq A \cap B$. Thus, $X \in \mathcal{P}(A \cap B)$, as required.

(c) $\mathcal{P}(A \times B)$ and $\mathcal{P}(A) \times \mathcal{P}(B)$

Neither of these sets is a subset of the other, and in fact they have no common elements. We can see that by analyzing the kinds of elements in each set:

- $\mathcal{P}(A \times B)$ contains subsets of $A \times B$, so these are sets containing ordered pairs whose first component is from A and whose second component is from B.
- $\mathcal{P}(A) \times \mathcal{P}(B)$ is the set of all ordered pairs whose first component is a *subset* of A and whose second component is a *subset* of B.

(3) (EO 2.7) Let $S = \mathcal{P}(T)$ be the power set of $T = \{1, 2, 3, 4\}$. Define a binary relation on S by XRY if either $X \subseteq Y$ or $Y \subseteq X$. Determine, for each property, whether the relation R is reflexive, symmetric, or transitive. Explain your answers.

This relation is reflexive because, for any $X \subseteq T$, $X = X$ so $X \subseteq X$.

This relation is symmetric because, if we take $X, Y \subseteq T$ and assume that XRY then we'll have that

$$\text{“either } X \subseteq Y \text{ or } Y \subseteq X\text{”}.$$

Since the ordering of an “or” statement doesn't change its truth (that is, $P \vee Q = Q \vee P$), we can rewrite this statement as

$$\text{“either } Y \subseteq X \text{ or } X \subseteq Y\text{”},$$

which is exactly the definition of YRX, as required.

This relation is not transitive. As a counterexample, consider $X = \{1\}, Y = \{1, 2\}$ and $Z = \{3\}$. Then XRY because $X \subseteq Y$, and YRZ because $Z \subseteq Y$, but $\sim (XRZ)$ because neither $X \subseteq Z$ nor $Z \subseteq X$.

(4) (EO 8) Let $\mathbb{N}^+$ denote the nonzero natural numbers. Define a binary relation R on $\mathbb{N}^+ \times \mathbb{N}^+$ by $(m, n)R(s, t)$ if $gcd(m, n) = gcd(s, t)$. Determine, for each property, whether the relation R is reflexive, symmetric, or transitive. Explain your answers.

This relation is reflexive: let $m, n \in \mathbb{N}^+$ be arbitrary fixed numbers, then $(m, n)R(m, n)$ because $gcd(m, n) = gcd(m, n)$.

This relation is symmetric because equality is symmetric. Let $m, n, s, t \in \mathbb{N}^+$ be arbitrary fixed numbers, then if $(m, n)R(s, t)$, it must be that $gcd(m, n) = gcd(s, t)$. By symmetry of $=$, we also have $gcd(s, t) = gcd(m, n)$ and hence $(s, t)R(m, n)$.

The relation is also transitive (again using properties of equality. Let $m, n, s, t, a, b \in \mathbb{N}^+$ and suppose $(m, n)R(s, t)$ and $(s, t)R(a, b)$. By definition, this means $gcd(m, n) = gcd(s, t)$ and $gcd(s, t) = gcd(a, b)$. Transitive of $=$ gives that $gcd(m, n) = gcd(a, b)$ and hence $(m, n)R(a, b)$, as required.

(5) (EO 12) Define an equivalence relation R on the positive integers $A = \{2, 3, 4, \ldots, 20\}$ by mRn if the largest prime divisor of m is the same as the largest prime divisor of n. How many equivalence classes does R have? List them.

Here are the blocks of the partition associated with the equivalence relation:

$$\{2, 4, 8, 16\} \qquad \{3, 6, 9, 12, 18\} \qquad \{5, 10, 15, 20\}$$

$$\{7, 14\} \qquad \{11\} \qquad \{13\} \qquad \{17\} \qquad \{19\}$$

Thus, there are eight equivalence classes (blocks).

(6) (EO 2.14) Let S be the set of composite integers n, $4 \leqslant n \leqslant 20$. Order S with the "divides" relation. Draw the Hasse diagram. List the minimal and the maximal elements.

By definition of composite numbers,

$$S = \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20\}$$

The minimal elements are the ones which have no factors in the set (because then no number is "smaller" than them according to the partial order). These are:

$$4, 6, 9, 10, 14, 15.$$

The maximal elements are those which don't divide any other element in the set.

$$20, 18, 16, 15, 14, 12$$

The Hasse diagram has the following chains:

$$4 - 8 - 16 \qquad 4 - 12 \qquad 4 - 20$$

$$6 - 12 \qquad 6 - 18$$

$$9 - 18 \qquad 10 - 20$$

Notice that $14, 15$ are each incomparable with every other element in this poset.

(7) (EO 2.16) Give an example of a poset with no maximal element.

Using the integers S as the base set consider the relation $\leqslant$, the usual ordering of numbers. This is a linear order so a maximal element would be a greatest element, but there is no greatest integer.