MATH CSE20 Homework 8 - Solutions

DUE Monday November 25

Assigned reading: SF Sections 1 and EO Section 2

(1) (SF 1.15) Which of the following are partitions of \{1, 2, \ldots, 8\}? Explain your answers.
   (a) \{\{1, 3, 5\}, \{1, 2, 6\}, \{4, 7, 8\}\}
       Not a partition because 1 appears in two blocks so the blocks aren’t disjoint.
   (b) \{\{1, 3, 5\}, \{2, 6, 7\}, \{4, 8\}\}
       Is a partition because the blocks are disjoint (have empty intersection) and each element of the set is in some block.
   (c) \{\{1, 3, 5\}, \{2, 6\}, \{2, 6\}, \{4, 7, 8\}\}
       Note that this set can equivalently be written as \{\{1, 3, 5\}, \{2, 6\}, \{4, 7, 8\}\}
       and now we see that each element of the original set appears in exactly one block. Thus, this is a partition.
   (d) \{\{1\}, \{2, 6\}, \{4, 8\}\}.
       Not a partition because 3 and 7 are in no blocks.

(2) (SF 1.20) Compare the following pairs of sets. Can they be equal? Is one a subset of the other? Can they have the same size (number of elements)?

(a) \(\mathcal{P}(A \cup B)\) and \(\mathcal{P}(A) \cup \mathcal{P}(B)\)

These sets are equal if \(A \subseteq B\) or \(B \subseteq A\). If this condition holds, then the two equal sets have the same size. To see that the subset condition guarantees equality, note that if \(A \subseteq B\) then \(A \cup B = B\) and \(\mathcal{P}(A) \subseteq \mathcal{P}(B)\) so \(\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(B)\). Similar arguments work in the case \(B \subseteq A\).

For general \(A, B\), \(\mathcal{P}(A \cup B) \supseteq \mathcal{P}(A) \cup \mathcal{P}(B)\) but we may not always have equality.

   - Let \(A, B\) be arbitrary. Then if \(X \in \mathcal{P}(A) \cup \mathcal{P}(B)\) then either \(X \subseteq A\) or \(X \subseteq B\). In either case, \(X \subseteq A \cup B\) so \(X \in \mathcal{P}(A \cup B)\).
   - To prove that the inclusion may be strict, consider the example \(A = \{1\}\) and \(B = \{2\}\). Then \(\mathcal{P}(A) = \{\emptyset, A\}, \mathcal{P}(B) = \{\emptyset, B\}\) so \(\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, 1, 2\}\). But, \(\mathcal{P}(A \cup B) = \mathcal{P}([1, 2]) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} \supset \{\emptyset, \{1\}, \{2\}\}\).

(b) \(\mathcal{P}(A \cap B)\) and \(\mathcal{P}(A) \cap \mathcal{P}(B)\)

These sets are equal (so, in particular, have the same size). To prove this, consider arbitrary sets \(A, B\). We need to prove subset inclusion in two directions.

   - To prove that \(\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)\), let \(X \in \mathcal{P}(A \cap B)\) so \(X \subseteq A \cap B\). By definition of intersection, this means that \(X \subseteq A\) and \(X \subseteq B\). Thus, \(X \in \mathcal{P}(A)\) and \(X \in \mathcal{P}(B)\). Hence, \(X \in \mathcal{P}(A) \cap \mathcal{P}(B)\), as required.
   - Conversely, to prove that \(\mathcal{P}(A \cap B) \supseteq \mathcal{P}(A) \cap \mathcal{P}(B)\), let \(X \in \mathcal{P}(A) \cap \mathcal{P}(B)\). Then \(X \in \mathcal{P}(A)\) and \(X \in \mathcal{P}(B)\). Thus, \(X \subseteq A\) and \(X \subseteq B\). In other words, every element of \(X\) is in both \(A\) and \(B\). This is
the definition of membership in the intersection, so we have that
\( X \subseteq A \cap B \). Thus, \( X \in \mathcal{P}(A \cap B) \), as required.

(c) \( \mathcal{P}(A \times B) \) and \( \mathcal{P}(A) \times \mathcal{P}(B) \)

Neither of these sets is a subset of the other, and in fact they have no
common elements. We can see that by analyzing the kinds of elements
in each set:
- \( \mathcal{P}(A \times B) \) contains subsets of \( A \times B \), so these are sets containing
  ordered pairs whose first component is from \( A \) and whose second
  component is from \( B \).
- \( \mathcal{P}(A) \times \mathcal{P}(B) \) is the set of all ordered pairs whose first component
  is a subset of \( A \) and whose second component is a subset of \( B \).

(3) (EO 2.7) Let \( S = \mathcal{P}(T) \) be the power set of \( T = \{1, 2, 3, 4\} \). Define a binary relation on \( S 
by XRY \) if either \( X \subseteq Y \) or \( Y \subseteq X \). Determine, for each property, whether the relation
\( R \) is reflexive, symmetric, or transitive. Explain your answers.

This relation is reflexive because, for any \( X \subseteq T \), \( X = X \) so \( X \subseteq X \).

This relation is symmetric because, if we take \( X, Y \subseteq T \) and assume that \( XRY \)
then we’ll have that

“either \( X \subseteq Y \) or \( Y \subseteq X \”).

Since the ordering of an “or” statement doesn’t change its truth (that is, \( P \lor Q =
Q \lor P \)), we can rewrite this statement as

“either \( Y \subseteq X \) or \( X \subseteq Y \”),

which is exactly the definition of \( YRX \), as required.

This relation is not transitive. As a counterexample, consider \( X = \{1\}, Y =
\{1, 2\} \) and \( Z = \{3\} \). Then \( XRY \) because \( X \subseteq Y \), and \( YRZ \) because \( Z \subseteq Y \),
but \( \sim (XRZ) \) because neither \( X \subseteq Z \) nor \( Z \subseteq X \).

(4) (EO 8) Let \( \mathbb{N}^+ \) denote the nonzero natural numbers. Define a binary relation \( R \) on
\( \mathbb{N}^+ \times \mathbb{N}^+ \) by \( (m, n)R(s, t) \) if \( \text{gcd}(m, n) = \text{gcd}(s, t) \). Determine, for each property, whether
the relation \( R \) is reflexive, symmetric, or transitive. Explain your answers.

This relation is reflexive: let \( m, n \in \mathbb{N}^+ \) be arbitrary fixed numbers, then
\( (m, n)R(m, n) \) because \( \text{gcd}(m, n) = \text{gcd}(m, n) \).

This relation is symmetric because equality is symmetric. Let \( m, n, s, t \in \mathbb{N}^+ \)
be arbitrary fixed numbers, then if \( (m, n)R(s, t) \), it must be that \( \text{gcd}(m, n) =
\text{gcd}(s, t) \). By symmetry of \( = \), we also have \( \text{gcd}(s, t) = \text{gcd}(m, n) \) and hence
\( (s, t)R(m, n) \).

The relation is also transitive (again using properties of equality. Let \( m, n, s, t, a, b \in
\mathbb{N}^+ \) and suppose \( (m, n)R(s, t) \) and \( (s, t)R(a, b) \). By definition, this means \( \text{gcd}(m, n) =
\text{gcd}(s, t) \) and \( \text{gcd}(s, t) = \text{gcd}(a, b) \). Transitive of \( = \) gives that \( \text{gcd}(m, n) =
\text{gcd}(a, b) \) and hence \( (m, n)R(a, b) \), as required.
(5) (EO 12) Define an equivalence relation \( R \) on the positive integers \( A = \{2, 3, 4, \ldots, 20\} \) by \( mRn \) if the largest prime divisor of \( m \) is the same as the largest prime divisor of \( n \). How many equivalence classes does \( R \) have? List them.

Here are the blocks of the partition associated with the equivalence relation:

\[
\{2, 4, 8, 16\} \quad \{3, 6, 9, 12, 18\} \quad \{5, 10, 15, 20\} \\
\{7, 14\} \quad \{11\} \quad \{13\} \quad \{17\} \quad \{19\}
\]

Thus, there are eight equivalence classes (blocks).

(6) (EO 2.14) Let \( S \) be the set of composite integers \( n, 4 \leq n \leq 20 \). Order \( S \) with the “divides” relation. Draw the Hasse diagram. List the minimal and the maximal elements.

By definition of composite numbers,

\[
S = \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20\}
\]

The minimal elements are the ones which have no factors in the set (because then no number is “smaller” than them according to the partial order). These are:

\[4, 6, 9, 10, 14, 15\].

The maximal elements are those which don’t divide any other element in the set.

\[20, 18, 16, 15, 14, 12\]

The Hasse diagram has the following chains:

\[
4 \rightarrow 8 \rightarrow 16 \quad 4 \rightarrow 12 \quad 4 \rightarrow 20 \\
6 \rightarrow 12 \quad 6 \rightarrow 18 \\
9 \rightarrow 18 \quad 10 \rightarrow 20
\]

Notice that 14, 15 are each incomparable with every other element in this poset.

(7) (EO 2.16) Give an example of a poset with no maximal element.

Using the integers \( S \) as the base set consider the relation \( \leq \), the usual ordering of numbers. This is a linear order so a maximal element would be a greatest element, but there is no greatest integer.