MATH CSE20 Homework 4
DUE Monday October 28

Assigned reading: Lo Section 2

(1) (Lo 2.2) Start with the statement “∀n ∈ N, if \( n^2 \) is even then \( n \) is even.” Which of the following statements say the same thing?
   (a) Every integer has an even square and is even.
       Not the same. This sentence asserts that every integer is even. The original statement does not.
   (b) If a given integer has an even square then that integer is even.
       Same. “a given . . .” is often a clue that a universal claim is being made.
   (c) For all integers, some will have an even square.
       Not the same, but the statement is true. This statement doesn’t give any guarantees about which integers will have even squares.
   (d) Any integer that has an even square will be even.
       Same.
   (e) If the square of some integer is even then it is even.
       Same.
   (f) All integers that are even have an even square.
       Not the same because it is the converse of the original statement.

(2) (Lo 2.3) For each of the following statements construct a statement of the form ∀ · · ·, if · · · then · · ·” that says the same thing.
   (a) Any correct algorithm, correctly coded, runs correctly.
       ∀ correct algorithms A, (A is correctly coded) ⇒ (A runs correctly).
   (b) Given any two odd integers, their product is odd.
       ∀ s, t ∈ \( \mathbb{Z} \), ((s odd) ∧ (t odd)) ⇒ (st odd).
   (c) Given any two integers whose product is odd, the integers themselves are odd.
       This is the converse of (b): ∀ s, t ∈ \( \mathbb{Z} \), (st odd) ⇒ ((s odd) ∧ (t odd)).

(3) (Lo 2.6) A number in \( \mathbb{R} - \mathbb{Q} \) is called irrational. Consider the statement “the product of any irrational number and any rational number is irrational”. Is the following proposed negation of the statement the correct negation? If not what is the correct negation?

   “There exists an irrational number \( x \) and an irrational number \( y \) such that the product \( xy \) is rational.”

Which is true, the original statement or its negation?

The proposed negation is incorrect. A correct version is “There exists an irrational number \( x \) and a rational number \( y \) such that the product \( xy \) is rational.”

The negation is true since we could take \( x \) to be any irrational number and \( y = 0 \). If we had instead started with “The product of any irrational number and any nonzero rational number is irrational,” then that statement
is true and its negation, “There exists an irrational number \( x \) and a nonzero rational number \( y \) such that the product \( xy \) is rational,” is false. The incorrect “negation” given in the problem is also true, for example if we take the irrational numbers \( x, y \) to both be equal to \( \sqrt{2} \).

(4) (Lo 2.7) Consider the statement “For all computer programs, \( P \), if \( P \) is correctly programmed then \( P \) compiles without warning messages.” What is the negation of this statement? Which is true, the original statement or its negation?

There exists a computer program \( P \) such that \( P \) is correctly programmed and \( P \) compiles with warning messages. Which is true depends on your interpretation of “correctly programmed.” Often a program will run just fine with warning messages. For example, many compilers give a warning message if a loop has no code in its body, but the empty body may be intentional.

(5) (Lo 2.19) Let \( P(x) \) and \( Q(x) \) be predicates with domain \( D \). For each pair of statement forms, state which are equivalent and explain your answer.

(a) \( \forall x \in D, (P(x) \land Q(x)) \) compared with \( (\forall x \in D, P(x)) \land (\forall x \in D, Q(x)) \).

**Equivalent statements:** If the first is true, then \( \forall x \in D P(x) \). Likewise, \( \forall x \in D P(x) \). Thus the second is true. Suppose the first is false, then there is \( x \in D \) such that either \( P(x) \) is false or \( Q(x) \) is false. If \( P(x) \) is false, then so is \( \forall x \in D P(x) \) and hence the second statement is false. If \( Q(x) \) is false, similar reasoning applies.

(b) \( \exists x \in D, (P(x) \land Q(x)) \) compared with \( (\exists x \in D, P(x)) \land (\exists x \in D, Q(x)) \).

**Not equivalent statements:** Let \( D = \mathbb{Z} \), let \( P(x) \) be “\( x \) is even” and let \( Q(x) \) be “\( x \) is odd.” Then the first statement is false and the second is true.

(c) \( \forall x \in D, (P(x) \lor Q(x)) \) compared with \( (\forall x \in D, P(x)) \lor (\forall x \in D, Q(x)) \).

**Not equivalent statements:** The example for (b) works here also.

(d) \( \exists x \in D, (P(x) \lor Q(x)) \) compared with \( (\exists x \in D, P(x)) \lor (\exists x \in D, Q(x)) \).

**Equivalent:** If the first is true, then there is some \( x \in D \) such that either \( P(x) \) is true or \( Q(x) \) is true. Hence either \( \exists x \in D P(x) \) is true or \( \exists x \in D P(x) \) is true. Thus the second statement is true. Suppose the first is false, then for all \( x \in D \), both \( P(x) \) and \( Q(x) \) are false. From this you can conclude that the second statement is false.

**Note:** You actually only need to do one of (a) and (d) and one of (b) and (c) because of negation. Negating both statements in (a) gives both statements in (d) with the predicates \( \sim P \) and \( \sim Q \) in place of the predicates \( P \) and \( Q \). Likewise for (b) and (c).

(6) Let \( D = \{ -48, -14, -8, 0, 1, 3, 16, 23, 26, 32, 36 \} \). Determine which of the following statements are true and which are false. Provide counterexamples for those statements that are false and justify the statements that are true.
(a) $\forall x \in D$, if $x$ is odd then $x > 0$.

**True** To prove this, we need to consider every element of $D$ and consider to whether the implication ($x$ is odd $\rightarrow$ ($x > 0$)) is true with this value of $x$.

- $x = -48$: the implication is vacuously true because $x$ is not odd.
- $x = -14$: the implication is vacuously true because $x$ is not odd.
- $x = -8$: the implication is vacuously true because $x$ is not odd.
- $x = 0$: the implication is vacuously true because $x$ is not odd.
- $x = 1$: the hypothesis of the implication is true (because $1$ is odd) and the conclusion is true (because $1 > 0$) so the implication is true.

In fact, for all remaining values $x \in D$, they conclusion of the implication will be T (because these values are all positive) so there’s no way to make the implication false (break its promise). Therefore, the implication is true at all remaining $x \in D$.

(b) $\forall x \in D$, if $x$ is less than 0 then $x$ is even.

**True** To prove this, we consider all the elements of $D$ where the hypothesis of the inner implication is true: $-48, -14, -8$. Each of these is even so the implication is true for each $x$ value in $D$.

(c) $\forall x \in D$, if $x$ is even then $x \leq 0$.

**False**: consider the counterexample $x = 16$ which is an even number in $D$ that is strictly positive.

(d) $\forall x \in D$, if the ones digit of $x$ is 6, then the tens digit is 1 or 2.

**False**: consider the counterexample $x = 36 \in D$.

(7) Write a negation for each statement. Bring the negation as deeply into the statement as possible.

*Note:* It will be very useful to remember the equivalence

$$p \rightarrow q = \neg p \lor q.$$

(a) For all real numbers $x$, if $x^2 \geq 1$ then $x > 0$.

Original: $\forall x \in \mathbb{R}, (x^2 \geq 1) \Rightarrow (x > 0)$

Negation: $\exists x \in \mathbb{R}, (\sim (x^2 \geq 1) \lor (x \leq 0))$

Which translates to: there exists a real number $x$ such that $x^2 \geq 1$ and $x \leq 0$. 

$$\exists x \in \mathbb{R}, (x^2 \geq 1) \land (x \leq 0)$$
(b) For all integers $d$, if $\frac{6}{d}$ is an integer then $d = 3$.

Original: \[ \forall d \in \mathbb{Z}, \ d \mid 6 \Rightarrow (d = 3) \]
\[ \forall d \in \mathbb{Z}, \sim (d \mid 6) \lor (d = 3) \]

Negation: \[ \sim ( \forall d \in \mathbb{Z}, \sim (d \mid 6) \lor (d = 3) ) \]
\[ \exists d \in \mathbb{Z}, \sim (\sim (d \mid 6) \lor (d = 3)) \]
\[ \exists d \in \mathbb{Z}, \ (d \mid 6) \land (d \neq 3) \]

Which translates to: There exists an integer $d$ such that $\frac{6}{d}$ is an integer and $d \neq 3$.

(c) For all real numbers $x$, if $x(x + 1) > 0$ then $x > 0$ or $x < -1$.

Original: \[ \forall x \in \mathbb{R}, \ x(x + 1) > 0 \Rightarrow ((x > 0) \lor (x < -1)) \]
\[ \forall x \in \mathbb{R}, \sim (x(x + 1) > 0) \lor ((x > 0) \lor (x < -1)) \]

Negation: \[ \sim (\forall x \in \mathbb{R}, \sim (x(x + 1) > 0) \lor ((x > 0) \lor (x < -1))) \]
\[ \exists x \in \mathbb{R}, \sim (\sim (x(x + 1) > 0) \lor ((x > 0) \lor (x < -1))) \]
\[ \exists x \in \mathbb{R}, \ x(x + 1) > 0 \land \sim ((x > 0) \lor (x < -1)) \]
\[ \exists x \in \mathbb{R}, \ x(x + 1) > 0 \land (x \leq 0) \land (x \geq -1) \]

Which translates to: There exists a real number $x$ such that $x(x + 1) > 0$ and $-1 \leq x \leq 0$.

(d) For all integers $a$, $b$, and $c$, if $a - b$ is even and $b - c$ is even then $a - c$ is even.

Original: \[ \forall a, b, c \in \mathbb{Z}, \ (a - b) \text{ is even} \land (b - c) \text{ is even} \Rightarrow (a - c) \text{ is even} \]
\[ \forall a, b, c \in \mathbb{Z}, \sim ((a - b) \text{ is even} \land (b - c) \text{ is even}) \lor (a - c) \text{ is even} \]

Negation: \[ \sim (\forall a, b, c \in \mathbb{Z}, \sim ((a - b) \text{ is even} \land (b - c) \text{ is even}) \lor (a - c) \text{ is even}) \]
\[ \exists a, b, c \in \mathbb{Z}, \sim ((a - b) \text{ is even} \land (b - c) \text{ is even}) \lor (a - c) \text{ is even} \]
\[ \exists a, b, c \in \mathbb{Z}, ((a - b) \text{ is even} \land (b - c) \text{ is even}) \land \sim ((a - c) \text{ is even}) \]
\[ \exists a, b, c \in \mathbb{Z}, ((a - b) \text{ is even} \land (b - c) \text{ is even}) \land (a - c) \text{ is odd} \]

Which translates to: There exists integers $a, b$, and $c$ such that $a - b$ is even, $b - c$ is even, and $a - c$ is odd.

(8) For each of the statements in the previous example, state whether the statement or its negation is true.

(a) For all real numbers $x$, if $x^2 \geq 1$ then $x > 0$. \textbf{False}

Negation: There exists a real number $x$ such that $x^2 \geq 1$ and $x \leq 0$. \textbf{True}

To prove the true existential statement, it’s enough to find an example: consider $x = -2$. Then $x^2 = 4 \geq 1$ but $x = -2 \leq 0$. 

(b) For all integers \( d \), if \( \frac{6}{d} \) is an integer then \( d = 3 \). \text{False}

Negation: There exists an integer \( d \) such that \( \frac{6}{d} \) is an integer and \( d \neq 3 \). \text{True}

To prove the true existential statement, it’s enough to find an example: consider \( d = 2 \). Then \( \frac{6}{2} = 3 \in \mathbb{Z} \) but \( d = 2 \neq 3 \).

(c) For all real numbers \( x \), if \( x(x + 1) > 0 \) then \( x > 0 \) or \( x < -1 \). \text{True}

Negation: There exists a real number \( x \) such that \( x(x + 1) > 0 \) and \( -1 \leq x \leq 0 \). \text{False}

(d) For all integers \( a, b, \) and \( c \), if \( a - b \) is even and \( b - c \) is even then \( a - c \) is even. \text{True}

Negation: There exists integers \( a, b, \) and \( c \) such that \( a - b \) is even, \( b - c \) is even, and \( a - c \) is odd. \text{False}