MATH CSE20 Test 1 Review Sheet

Test Tuesday October 15 in lecture: CENTER 115, 3:30pm

Textbook sections: Unit BF Sections 1 and 2

(1) All questions from Homeworks 1 and 2.

(2) (BF Review Question 5) Which of the following functions is the constant 1 function?
    A constant 1 function would have all 1s in the rightmost column of the truth-table.
    (a) $\sim p \lor (p \land q)$
        NOT a constant 1 function because:
        \[
        \begin{array}{c|c|c|c}
        p & q & \sim p & \sim p \lor (p \land q) \\
        \hline
        1 & 1 & 0 & 1 \\
        1 & 0 & 0 & 0 \\
        0 & 1 & 1 & 0 \\
        0 & 0 & 1 & 1 \\
        \end{array}
        \]
    (b) $(p \land q) \lor (\sim p \lor (p \land q))$
        IS a constant 1 function because:
        \[
        \begin{array}{c|c|c|c|c|c}
        p & q & p \land q & \sim p & \sim p \lor (p \land q) & (p \land q) \lor (\sim p \lor (p \land q)) \\
        \hline
        1 & 1 & 1 & 0 & 0 & 1 \\
        1 & 0 & 0 & 0 & 1 & 1 \\
        0 & 1 & 0 & 1 & 1 & 1 \\
        0 & 0 & 1 & 1 & 1 & 1 \\
        \end{array}
        \]
    (c) $(p \land \neg q) \land (\neg p \lor q)$
        NOT a constant 1 function because:
        \[
        \begin{array}{c|c|c|c|c|c|c|c}
        p & q & \neg q & p \land \neg q & (q \lor \sim p) & (p \land \neg q) \land (\neg p \lor q) \\
        \hline
        1 & 0 & 1 & 1 & 0 & 0 \\
        1 & 1 & 0 & 0 & 1 & 0 \\
        0 & 0 & 1 & 0 & 1 & 0 \\
        0 & 1 & 0 & 0 & 1 & 0 \\
        \end{array}
        \]
    (d) $((\neg p \land q) \land (q \land r)) \land \neg q$
        NOT a constant 1 function because:
        \[
        \begin{array}{c|c|c|c|c|c|c|c|c|c}
        p & q & \sim p & \sim q & r & \sim p \land q & q \land r & (\sim p \land q) \land (q \land r) \land \sim q \\
        \hline
        1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
        1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
        1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
        1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
        0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
        0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
        0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
        0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
        \end{array}
        \]
(e) \((\neg p \lor q) \lor (p \land q)\)

**NOT** a constant 1 function because:

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<tr>
<th>(p)</th>
<th>(q)</th>
<th>(\sim p)</th>
<th>(\sim p \lor q)</th>
<th>(p \land q)</th>
<th>((\sim p \lor q) \lor (p \land q))</th>
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</thead>
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(3) (Epp 1.1.52) The symbol \(\oplus\) is used to denote exclusive or. We can see that \(p \oplus q = (p \lor q) \land \neg(p \land q)\).

(a) Find simpler formulas that are equal to \(p \oplus p\) and \((p \oplus p) \oplus p\).

It’s given that \(p \oplus q = (p \lor q) \land \sim(p \land q)\)

So \(p \oplus p = (p \lor p) \land \sim(p \land p)\)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>((p \lor p) \land \sim(p \land p))</td>
<td>1. Given</td>
</tr>
<tr>
<td>(p \land \sim(p \land p))</td>
<td>2. (p \lor p = p), by the Indempotent Law</td>
</tr>
<tr>
<td>(p \land \sim p)</td>
<td>3. (p \land p = p), by the Indempotent Law</td>
</tr>
<tr>
<td>(c)</td>
<td>4. (p \land \sim p) is a contradiction, by the Negation Law</td>
</tr>
</tbody>
</table>

\(p \oplus p\) simplifies to \(c\), or in other words, \(p \oplus p\) is always false.

Next, we simplify \((p \oplus p) \oplus p\)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
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</thead>
<tbody>
<tr>
<td>((p \lor p) \land \sim((p \lor p) \land p))</td>
<td>1. Given</td>
</tr>
<tr>
<td>((c \lor p) \land \sim(c \land p))</td>
<td>2. (p \oplus p = c), as shown above</td>
</tr>
<tr>
<td>(p \land \sim(c \land p))</td>
<td>3. (c \lor p = p) by the Identity Law</td>
</tr>
<tr>
<td>(p \land \sim c)</td>
<td>4. (c \land p = c), by the Universal Bound Law</td>
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<tr>
<td>(p \land t)</td>
<td>5. (\sim c = t), by negation of (c)</td>
</tr>
<tr>
<td>(p)</td>
<td>6. (p \land t = p), by the Identity Law</td>
</tr>
</tbody>
</table>

\((p \oplus p) \oplus p\) simplifies to \(p\).

(b) Is \((p \oplus q) \oplus r\) equal to \(p \oplus (q \oplus r)\)? Justify your answer.

One approach is to expand both propositions, and then simplify, to check for same-ness. That could get rather complicated. A quicker approach would be to check the truth values of \((p \oplus q) \oplus r\) and \(p \oplus (q \oplus r)\), as two Boolean functions are equal if they have the same truth tables.
Consider the two Boolean functions

\[ p \land (q \lor \sim p) \quad \sim (\sim p \lor \sim q) \]

(a) Prove that they are equal by filling in the following step-by-step truth table. Make sure to label each column you use. You may not need all columns.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( \sim p )</th>
<th>( \sim q )</th>
<th>( q \lor \sim p )</th>
<th>( \sim (\sim p \lor \sim q) )</th>
<th>( p \land (q \lor \sim p) )</th>
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(b) Write the name of the property that justifies each step of the following proof sequence demonstrating that the two Boolean functions are equal.

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons (name of property)</th>
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</thead>
<tbody>
<tr>
<td>1. ( p \land (q \lor \sim p) )</td>
<td>Given</td>
</tr>
<tr>
<td>2. ( (p \land q) \lor (p \land \sim p) )</td>
<td>Distributive Law on 1.</td>
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<tr>
<td>3. ( (p \land q) \lor c )</td>
<td>Negation Law on 2.</td>
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<td>4. ( (p \land q) )</td>
<td>Identity Law on 3.</td>
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<td>5. ( \sim (\sim (p \land q)) )</td>
<td>Double Negation Law on 4.</td>
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<td>6. ( \sim (\sim p \lor \sim q) )</td>
<td>DeMorgan’s Law on 5.</td>
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</tbody>
</table>

(5) Verify Theorem 2 on page BF-6 by proving that each of the rules are true. For example, prove that \( (p \land q) \land r = p \land (q \land r) \).

- Associative Rule: \( (p \lor q) \lor r = p \lor (q \lor r) \)

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<th>( q )</th>
<th>( r )</th>
<th>( p \lor q )</th>
<th>( q \lor r )</th>
<th>( (p \lor q) \lor r )</th>
<th>( p \lor (q \lor r) )</th>
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</table>
• Distributive Rules: \( p \land (q \lor r) = (p \land q) \lor (p \land r) \)

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<thead>
<tr>
<th></th>
<th></th>
<th>p \land q</th>
<th>p \lor r</th>
<th>(p \land q) \lor (p \land r)</th>
<th>p \land (q \lor r)</th>
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• Indempotent Rules: \( p \land p = p \) \( p \lor p = p \)

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<tr>
<th></th>
<th>p \land p</th>
<th>p \lor p</th>
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<tbody>
<tr>
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• Double Negation: \( \neg (\neg p) = p \)

<table>
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<th></th>
<th>\neg p</th>
<th>\neg (\neg p)</th>
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• DeMorgan’s Rules: \( \neg (p \land q) = \neg p \lor \neg q \)

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<th></th>
<th>p</th>
<th>q</th>
<th>p \land q</th>
<th>\neg p</th>
<th>\neg q</th>
<th>\neg (p \land q)</th>
<th>\neg p \lor \neg q</th>
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*This rule is extremely important.*

• Commutative Rules: \( p \land q = q \land p \)

<table>
<thead>
<tr>
<th></th>
<th>p</th>
<th>q</th>
<th>p \land q</th>
<th>q \land p</th>
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• Absorption Rules: \( p \lor (p \land q) = p \)

<table>
<thead>
<tr>
<th></th>
<th>p</th>
<th>q</th>
<th>p \land q</th>
<th>p \lor (p \land q)</th>
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• Bound Rules: \( p \land 0 = 0 \) \( p \land 1 = p \)
• Negation Rules: $p \land (\sim p) = 0$ $p \lor (\sim p) = 1$

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\sim p$</th>
<th>$p \land (\sim p)$</th>
<th>$p \lor (\sim p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
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</table>

(6) Write a Boolean function in DNF that is equivalent to the output column of the following truth-table.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$r$</th>
<th>output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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DNF: $(p \land q \land \sim r) \lor (p \land \sim q \land \sim r) \lor (\sim p \land q \land \sim r)$

(7) (Spring 2011, Midterm 1) Convert the following as indicated:

a. Convert octal $102_8$ to decimal.

$8^2(1) + 8^1(0) + 8^0(2) = 64 + 0 + 2 = 66_{10}$

b. Convert $B7C5_{16}$ to octal.

You can do this 2 ways:

1. Convert $B7C5_{16}$ to decimal, and then to octal

2. Convert $B7C5_{16}$ to binary, then octal

We will be using the second method, as it is quicker. Since 16 and 8 are both powers of 2, we can convert using binary. First convert $B7C5_{16}$ to binary, one character at a time.

$B = 11$ $7 = 0111$ $C = 12$ $5 = 0101$

\[
\begin{array}{c}
(1011) \\
(0111) \\
(1100) \\
(0101)
\end{array}
\]
Each hexadecimal symbol represents one of $2^4$ things, and these can be described by a group of four bits. To convert to octal, group these 1’s and 0’s into groups of 3, since each octal symbol represents one of $2^3$ things.

\[
\begin{array}{cccccc}
(001) & (011) & (011) & (111) & (000) & (101) \\
1 & 3 & 3 & 7 & 0 & 5
\end{array}
\]

Then string each number together, and you are finished. $B7C5_{16} = 1337058_{10}$

(8) (Spring 2011, Midterm 1) Find the range (minimum and maximum numbers that can be represented) of the following number system: a two’s complement system with 10 bits.

First, note that $n$ bits can represent, at maximum, $2^n$ objects. That means you can represent numbers 0 to $2^n - 1$ (the -1 is there to account for the representation of 0).

However, this is a Two’s Complement System, meaning that of $n$ bits, $n - 1$ are used to represent digits. The leftmost bit/most significant bit represents sign. With $n$ bits, the range of number that can be represented is $-(2^{n-1})$ to $(2^{n-1} - 1)$.

So for 10 bits, you get a range of $-(2^9)$ to $(2^9 - 1)$, or [-512 to 511].

(9) (Spring 2011, Midterm 1) We have defined and learned the idea of two’s complement for $n$-bit binary numbers. Given an $n$-digit system with base 8, define the eight’s complement representation.

Then, show the arithmetic of $-x + y$ where $x = 216_8$ and $y = 65_8$ with a 7-digit system in eight’s complement representation.

The eight’s complement representation of $-x$ in this system would be the octal form of $8^n - x$. So, to compute $-x + y$ where $x = 216_8$ and $y = 65_8$ using a 7-digit system, we calculate

\[
8^7 - 2 \cdot 8^2 - 1 \cdot 8^1 - 6 \cdot 8^0 = 2097010_{10} = 7 \cdot 8^6 + 7 \cdot 8^5 + 7 \cdot 8^4 + 7 \cdot 8^3 + 5 \cdot 8^2 + 6 \cdot 8^1 + 2 \cdot 8^0 = 7777562_8
\]

Then

\[-x + y = 7777562_8 + 65_8 = 7777647_8\]

(10) (Epp 1.5.27-30) Find the decimal representations for the integers with the following 8-bit (two’s complement) representations.

(a) 11010011_2

There are two approaches. First note that since the most significant bit (the 8th bit) is 1, the decimal representation is negative.

- Method 1 is to flip all the bits and then add 1.

\[
\begin{array}{c}
00101100 \quad \text{(flip all the bits)} \\
+ \quad 1 \quad \text{(add 1)} \\
\hline
00101101
\end{array}
\]

Translate this to decimal: \(2^5 + 2^3 + 2^2 + 2^0 = 32 + 8 + 4 + 1 = 45\). Since it is negative, the answer is \([-45]\).

- Method 2: Translate 11010011 to decimal.

\[128 + 64 + 16 + 2 + 1 = 211.\]

To get the answer, recall that \(211 = 2^n - x\), where \(n\) is the number of bits. Solving for \(x\) in \(211 = 2^{10} - x\), you get \([-45]\). Both methods are equivalent.

(b) 10011001\textsubscript{2}

- Method 1:

\[
\begin{array}{c}
01100110 \quad \text{(flip bits)} \\
+ \quad 1 \quad \text{(add 1)} \\
\hline
01100111
\end{array}
\]

Translate this to decimal: \(64 + 32 + 4 + 2 + 1 = 103\). Since the MSB of the original number was 1, the answer is negative, so \([-103]\).

(c) 11110010\textsubscript{2}

- Method 2:

Translate to decimal: \(128 + 64 + 32 + 16 + 2 = 242\). Since the MSB was 1, the integer representation is \([-14]\).

(d) 10111010\textsubscript{2}

- Method 2:

\[
\begin{array}{c}
01000101 \quad \text{(flip bits)} \\
+ \quad 1 \quad \text{(add 1)} \\
\hline
01000110
\end{array}
\]

Translate to decimal: \(64 + 4 + 2 = 70\). Since the MSB is 1, the representation is negative, so \([-70]\).

(11) The NAND function has the truth table and logic gate:

<table>
<thead>
<tr>
<th>(A)</th>
<th>(B)</th>
<th>(Y = A \ \text{NAND} \ B)</th>
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</tbody>
</table>

Explain why any digital circuit can be rewritten using only NAND gates.

All logic circuits can be written as combinations of the AND, OR, NOT gates. If these 3 gates can be rewritten using NAND gates, then all logic circuits can
be written using NAND gates.

- **NOT Gate**

  \[
  A \quad | \quad x = A \text{ NAND } A \quad | \quad \sim A
  \]

  \[
  \begin{array}{c|c|c}
  1 & 0 & 0 \\
  0 & 1 & 1 \\
  \end{array}
  \]

  \[
  \sim A = A \text{ NAND } A
  \]

- **AND Gate**

  \[
  A \quad | \quad B \quad | \quad A \text{ NAND } B \quad | \quad Y = (A \text{ NAND } B) \text{ NAND } (A \text{ NAND } B) \quad | \quad A \land B
  \]

  \[
  \begin{array}{c|c|c|c|c}
  1 & 1 & 0 & 1 & 1 \\
  1 & 0 & 1 & 0 & 0 \\
  0 & 1 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 \\
  \end{array}
  \]

  \[
  A \land B = (A \text{ NAND } B) \text{ NAND } (A \text{ NAND } B)
  \]

- **OR Gate**

  \[
  A \quad | \quad B \quad | \quad A \text{ NAND } A \quad | \quad B \text{ NAND } B \quad | \quad (A \text{ NAND } A) \text{ NAND } (B \text{ NAND } B) \quad | \quad A \lor B
  \]

  \[
  \begin{array}{c|c|c|c|c|c}
  1 & 1 & 0 & 0 & 1 & 1 \\
  1 & 0 & 0 & 1 & 1 & 1 \\
  0 & 1 & 1 & 0 & 1 & 1 \\
  0 & 0 & 1 & 1 & 0 & 0 \\
  \end{array}
  \]

  \[
  A \lor B = (A \text{ NAND } A) \text{ NAND } (B \text{ NAND } B)
  \]