Showing a System is Linear and Shift Invariant

Prepared by: Piotr Dollar

1 Showing Linearity

To show a system H is linear, we need to show that for all functions f_1 and f_2, H satisfies the following equation:

$$H[\alpha f_1(x) + \beta f_2(x)] = \alpha H[f_1(x)] + \beta H[f_2(x)]$$

That is we need to show the left side equals the right side in the above equation. How does one do this?

1. Find $H[\alpha f_1(x) + \beta f_2(x)]$. To do this, let $r(x) = \alpha f_1(x) + \beta f_2(x)$, find $H[r(x)]$ based on the definition of H, then substitute $f_1(x)$ and $f_2(x)$ back in. We will do some examples below if this is not clear.
2. Find $\alpha H[f_1(x)] + \beta H[f_2(x)]$.
3. If $H[\alpha f_1(x) + \beta f_2(x)] = \alpha H[f_1(x)] + \beta H[f_2(x)]$ then the system is linear, otherwise it is not.

Example 1: $H[f(x)] = f(2x)$

1. Let $r(x) = \alpha f_1(x) + \beta f_2(x)$;
   $H[r(x)] = r(2x) = \alpha f_1(2x) + \beta f_2(2x)$.
2. $\alpha H[f_1(x)] + \beta H[f_2(x)] = \alpha f_1(2x) + \beta f_2(2x)$
3. $H[\alpha f_1(x) + \beta f_2(x)] = \alpha H[f_1(x)] + \beta H[f_2(x)]$,
   so this system is **Linear**!

Example 2: $H[f(x)] = (f(x))^2$

1. Again, let $r(x) = \alpha f_1(x) + \beta f_2(x)$;
   $H[r(x)] = (r(x))^2 = (\alpha f_1(x) + \beta f_2(x))^2$
2. $\alpha H[f_1(x)] + \beta H[f_2(x)] = \alpha(f_1(x))^2 + \beta((f_2(x))^2$
3. $(\alpha f_1(x) + \beta f_2(x))^2 \neq \alpha(f_1(x))^2 + \beta((f_2(x))^2$,
   so this system is **NOT Linear**.

2 Showing Shift Invariance

This is sometimes referred to as **time invariance** or **spatial invariance** or a **fixed parameter system**.

Showing a system is shift invariance follows a very similar process to showing that it is linear. We need to show for all functions f, if: $g(x) = H[f(x)]$ then the following holds:

$$g(x + x_0) = H[f(x + x_0)]$$

So, again, how is this done?

1. Find $g(x + x_0)$. Do this simply by plugging in $x + x_0$ into our equation for $g(x)$. This may be confusing because you're plugging in $x + x_0$ for x! Hopefully the examples below will clarify.
2. Find $H[f(x + x_0)]$. The important thing to realize is that $f(x + x_0)$ is only a function of x. So anything H does to $f(x + x_0)$ it only does to x and not x_0. If this is not clear, let $r(x) = f(x+x_0)$, find $H[r(x)]$ and then expand. Again examples below will clarify.
3. If $g(x + x_0) = H[f(x + x_0)]$ then the system is shift invariant, otherwise it is not.

Example 1: $H[f(x)] = f(2x) = g(x)$

1. $g(x + x_0) = f(2(x + x_0)) = f(2x + 2x_0)$
2. $H[f(x + x_0)] = f(2x + x_0)$.
   Note that H doubles the x – but not the x_0 (which is just a constant). If this still isn't clear, let $r(x) = f(x + x_0)$. Now $H[r(x)] = r(2x)$ and $r(2x) = f(2x + x_0)$.
3. $g(x + x_0) \neq H[f(x + x_0)]$ so the system is **NOT shift invariant**.

Example 2: $H[f(x)] = (f(x))^2 = g(x)$

1. $g(x + x_0) = (f(x + x_0))^2$
2. $H[f(x + x_0)] = (f(x + x_0))^2$.
   To see why, let $r(x) = f(x + x_0)$. Now $H[r(x)] = (r(x))^2 = (f(x + x_0))^2$.
3. $g(x + x_0) = H[f(x + x_0)]$ so the system is **shift invariant**.