Problem 1 Let $A$ be the set $\{a, b\}$ and $B$ be the set $\{1, 2, 3\}$.

a. How many functions $f: A \rightarrow B$ are there? (In other words, how many distinct functions exist mapping from $A$ to $B$?)

b. How many functions $f: A \rightarrow B$ are there that are onto?

c. How many functions $f: B \rightarrow A$ are there?

d. How many functions $f: B \rightarrow A$ are there that are onto?

Be sure to explain your reasoning. For the purpose of this question, we assume that functions are defined on their entire domain.

Problem 2 The Fibonacci sequence is defined as follows: $F_0 = 0$, $F_1 = 1$, and, for $n > 1$, $F_n = F_{n-2} + F_{n-1}$. The first few elements of the sequence are 0, 1, 1, 2, 3, 5, 8, 13, ... 

Let $\phi$ represent the “golden ratio”: $\phi = \frac{1 + \sqrt{5}}{2} \approx 1.618033989$.

Prove, by induction, that for all $n \geq 1$ we have $F_n \leq \phi^{n-1}$.

Hint: $1 + \phi = \phi^2$, as you can easily verify.

Problem 3 In this problem, you will design DFAs that recognize numbers that are congruent to 1 modulo 5, i.e., that are equal to 1 more than some multiple of 5. For example, $11 = 2 \cdot 5 + 1$, so 11 (appropriately encoded) should be accepted by your DFAs, whereas $31337 = 6267 \cdot 5 + 2$, so 31337 should not be accepted by your DFAs. In part a. the input to your DFA will be encoded in unary notation; in part b. the input will be encoded in binary.

a. For the alphabet $\{1\}$, describe a DFA that recognizes the language of all strings that, interpreted in unary notation, are one more than a multiple of 5. In unary notation, 11 is encoded as 1111111. 

b. For the alphabet $\{0, 1\}$, describe a DFA that recognizes the language of all strings that, interpreted as numbers in binary notation (with least significant bit last), are one more than a multiple of 5. In binary notation, 11 is encoded as 1011, 31337 as 111101001101001. (It is okay for numbers to be represented with leading zeroes.)

For both parts, it is sufficient to draw the figure representing your DFAs; you need not give the formal 5-tuple specification.

Note: Five states suffice for the DFAs in both parts.
Problem 4 For Machine $M_1$ described in Figure 1:

a. Give the formal specification of Machine $M_1$ as a 5-tuple $M_1 = (Q, \Sigma, \delta, q_0, F)$.

b. Describe, as succinctly as you can, the language recognized by Machine $M_1$.

Problem 5 Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA. Provided that $L(M) \neq \emptyset$, define $\mu(M)$ to be the length of the shortest string $w$ accepted by $M$. For example, if $\mu(M) = 5$ then $M$ accepts at least one string of length 5 but does not accept any string of length 4 or less.

a. Give an algorithm, SHORTEST_ACCEPTING, that, given the description of a DFA $M$, computes and outputs $\mu(M)$.

b. Prove that, for $M = (Q, \Sigma, \delta, q_0, F)$, $\mu(M) \leq |Q|$ whenever $L(M) \neq \emptyset$. (Here $|Q|$ is the number of elements in the set $Q$.)