

December 2, 2008

Plan to show your work as much as possible (so that appropriate partial credit can be given). The Final will be a a CLOSED BOOK test. However, you may have two 8 1/2 by 11 inch sheets of paper with hand-written notes on them, if you wish. Answers can be left in unexpanded form, i.e., using fractions, ! and $\binom{x}{y}$. Calculators can be used but shouldn't be necessary.

1. If $f(x) = \frac{\sqrt{x}(x^3+x)}{x^2-10}$ and $g(x) = x^2 \log(x)$, then select all the answers that apply:

(a) $f(x) = O(g(x))$;

(b) $f(x) = o(g(x))$;

(c) $g(x) = \Omega(f(x))$;

(d) $f(x) = \Omega(g(x))$;

(e) $g(x) = O(f(x))$.

2. Prove by induction that for all $n \geq 1$,

$$1 + \frac{1}{2^{1/3}} + \frac{1}{3^{1/3}} + \dots + \frac{1}{n^{1/3}} \geq n^{2/3}$$

3. In how many ways can 20 identical jellybeans be distributed to 2 boys and 3 girls if every girl is to get **at least** 2 jellybeans, and every boy is to get **at most** 2 jellybeans?

4. Six light bulbs are selected at random from a set of 10 light bulbs of which 4 are defective. What is the **expected number** of defective bulbs selected?

5. In how many ways can six people be assigned to three (nonempty) committees?

6. A password is to be created using the 26 letters from A to Z. How many possible 5-letter passwords are there which **don't have** both O and Q?

7. A biased coin has probability $2/3$ of landing Heads and $1/3$ of landing Tails. You flip the coin n times. What is the **expected number** of occurrences of 2 consecutive Heads (i.e., the number of occurrences of HH in the sequence of n flips; thus, HHH has 2 occurrences of HH)?

8. Let $A(n)$ denote the number of strings of length n made up of the 3 symbols $\{0, 1, 2\}$ which **do not have** two consecutive 2's. Thus, $A(1) = 3$, $A(2) = 8$.

(a) Show that $A(n)$ satisfies the recurrence $A(n) = 2A(n-1) + 2A(n-2)$ for $n \geq 3$.

(b) Find an explicit expression for $A(n)$ (i.e., solve the above recurrence).

9. An urn contains 3 Red balls and 4 White balls. A random ball B_1 is drawn from the urn. If B_1 is Red then the ball is replaced along with 2 additional Red and 4 additional White balls. On the other hand, if B_1 is White then the ball is replaced along with 1 additional White ball. Now, a second ball B_2 is randomly drawn from the urn.

(a) What is the probability that B_2 is White?

(b) What is the probability that B_1 is White **given that B_2 is Red**?

10. In how many ways can 4 boys and 3 girls sit in a row if every boy is seated next to **at least one** girl?

11. An urn contains 5 Red marbles and 1 Blue marble. You repeatedly draw out a random marble (without replacement) until you get the Blue marble. What is the expected number of draws until the Blue marble is first drawn?