Problem 1: NP, RP and BPP

(a) Show that if \(L_1, L_2 \in \text{RP}\) (resp. \(\text{BPP}\)) then \(L_1 \cup L_2 \in \text{RP}\) and \(L_1 \cap L_2 \in \text{RP}\) (resp. \(\text{BPP}\)).

(b) In class, we defined the class \(\text{ZPP} = \text{RP} \cap \text{coRP}\). Show that if \(\text{coNP} \subseteq \text{RP}\) then \(\text{NP} = \text{ZPP}\).

Problem 2: More on NP, RP and BPP

(a) In class we defined class \(\text{BPP}\) as follows: \(\text{BPP}\) is the class of all languages \(L\) for which there exists a probabilistic polynomial-time TM \(M\), such that:

\[
x \in L \Rightarrow \Pr_r [M(x,r) = 1] \geq 1 - \epsilon \\
x \notin L \Rightarrow \Pr_r [M(x,r) = 1] \leq \epsilon
\]

where the error probability is \(\epsilon = 1/3\). We also observed that the error probability can be reduced to \(1/2^n\) using standard repetition techniques. (Formally, we proved the result for \(\text{RP}\). You can read the details of a similar proof for \(\text{BPP}\) from the textbook.) In other words, the class \(\text{BPP}\) remains the same even if we replace the \(\epsilon = 1/3\) in the above definition with \(\epsilon = 1/2^n\).

- Show that if \(L\) is \(\text{NP}\)-complete and \(L \in \text{RP}\), then \(\text{NP} \subseteq \text{RP}\).
- Show that if \(\text{NP} \subseteq \text{BPP}\) then \(\text{NP} = \text{RP}\).

(b) Let \(\mathcal{BPP}^{\mathcal{BPP}} = \bigcup_{L \in \mathcal{BPP}} \mathcal{BPP}^L\) (the exponent \(L\) denotes oracle access to a language \(L \in \mathcal{BPP}\)). Show that \(\mathcal{BPP}^{\mathcal{BPP}} = \mathcal{BPP}\). Can you prove the same result for \(\text{RP}\)?