10 points/problem

SOLUTION FOR FINAL EXAM

60

ECE 53A Fall 2007, Final, December 13, 2007, Name Solution

1. A battery has a rating of 12V and 10AH. The battery is used to supply a portable computer which consumes 30W. How long can the computer function on a fully charged battery?

$$\text{Time} = \frac{12 \times 10 \ V \times A \cdot H}{30 W} = 4 \ Hrs$$

10 points: If the answer was wrong, partial credits were awarded based on the formula and correctness of approach

2. A copper wire has dimensions of 1um thickness, 1um width, and 100um length. The resistivity of the copper is about 2uohm-cm.

a. Derive the resistance of the wire.

$$r = \frac{2 \times 10^{-6} \ \Omega\text{-cm}}{1 \times 10^{-6} \times 1 \times 10^{-6} \ m^2} \times 100 \times 10^{-6} \ m = \frac{2 \times 10^{-6}}{10^{-6}} \times 100 \times \frac{cm \ m}{100 cm} \cdot \frac{m}{m^2}$$

$$= 2 \Omega$$

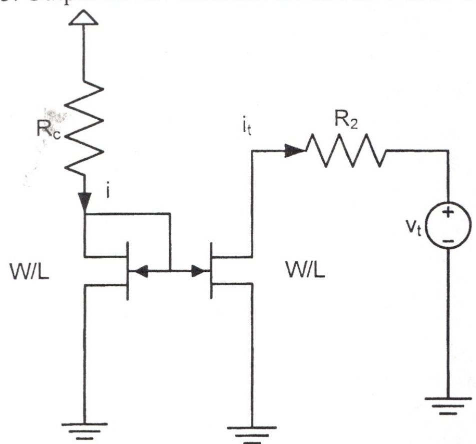
b. Suppose the wire carries 0.1mA (0.0001A) current. Derive the voltage drop across the wire and power consumption on the wire.

$$\Delta V = r \times I = 2 \times 0.0001 = 0.0002 \ V.$$

$$P = rI^2 = 2 \times (10^{-4})^2 = 2 \times 10^{-8} \ W.$$

10 points: 5 points for part A and 5 points for part B.
If the answer was wrong, partial credits were awarded based on the formula and correctness of approach.
In part B, 2 points were taken off if one of the values asked for is wrong or ignored. (Voltage or Power)

3. Output driver: Consider the current mirror circuit below.



a. Determine the output current i_t as a function of input current i assuming both MOSFETs operate under the saturation region.

$i = -i_L$ ie. $i_L = -i$

$\underline{i_L = -i}$

6 points: If the answer was wrong, partial credits were awarded based on of approach

b. Determine the range of input current i and output voltage v_t for which both MOSFETs operate in the saturation region.

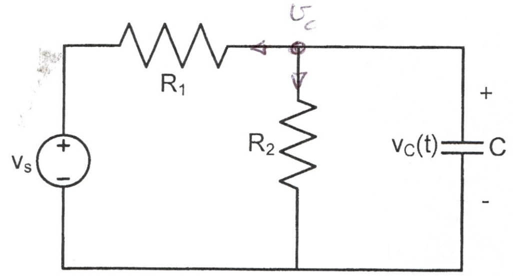
① $V_g = V_{dd} - R_c i \geq V_T \Rightarrow \underline{i \leq \frac{V_{dd} - V_T}{R_c}}$

② $V_D \geq V_g - V_T \Rightarrow V_t - R_2 i \geq V_{dd} - R_c i$

$\Rightarrow \underline{V_t \geq V_{dd} + (R_2 - R_c) i}$

4 points: If the answer was wrong, partial credits were awarded based on of approach. In addition to simply writing down the general NMOS relations, the voltages and current values from the given circuit needs to be substituted for receiving credits.

4. A first-order leakage RC line modeled below has two resistors and a capacitor C. Assume that the voltage source is a step function, i.e. $v_s(t)=u(t)$, i.e. $v_s(t)= 0$ for $t< 0$, $v_s(t)= 1$ for $t>= 0$.



a. Derive the state equation.

KCL or NODAL EQ at NODE $v = v_c$ =

For $t \geq 0$, $\frac{v_c - v_s}{R_1} + \frac{v_c}{R_2} + C\frac{dv_c}{dt} = 0$

$$\frac{dv_c}{dt} = -\left(\frac{1}{R_1C} + \frac{1}{R_2C}\right) v_c + \frac{1}{R_1C} v_s$$
$$= -\frac{1}{C}\left(\frac{R_1+R_2}{R_1R_2}\right) v_c + \frac{1}{R_1C} \quad (v_s = 1 \text{ for } t \geq 0)$$

Let $x \equiv v_c$

$$\dot{x} = -\frac{1}{C}\left(\frac{R_1+R_2}{R_1R_2}\right) x + \frac{1}{R_1C}$$

b. Derive the solution of the capacitor voltage $v_C(t)$ with an initial voltage $v_C(0)= 0V$.

$$\dot{x} = ax + \frac{1}{R_1C} \qquad a \equiv -\frac{1}{C}\left(\frac{R_1+R_2}{R_1R_2}\right)$$

$$x(t) = v_c(t) = Ke^{+at} - \frac{R_2C}{R_1+R_2}$$

$x_p = A \qquad aA + \frac{1}{R_1C} = 0 \qquad A = -\frac{1}{R_1C\,a} = -\frac{1}{R_1C}\left(\frac{CR_1R_2}{R_1+R_2}\right) = -\frac{R_2C}{R_1+R_2}$

$$x(0) = v_c(0) = 0 = K\cancel{e^{-at}} - \frac{R_2C}{R_1+R_2} \qquad K = \frac{R_2C}{R_1+R_2}$$

$$x(t) = v_c(t) = \frac{R_2C}{R_1+R_2}e^{+at} - \frac{R_2C}{R_1+R_2} \quad \text{where} \quad a = -\frac{1}{C}\left(\frac{R_1+R_2}{R_1R_2}\right)$$

c. Derive the solution of the capacitor voltage $v_C(t)$ with an initial voltage $v_C(0)= 0.5V$.

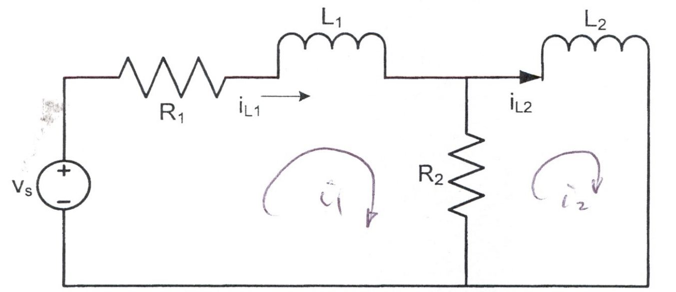
$$x(0) = v_c(0) = 0.5 = K\cancel{e^{-at}} - \frac{R_2C}{R_1+R_2}$$

$$K = 0.5 + \frac{R_2C}{R_1+R_2}$$

$$x(t) = v_c(t) = \left[0.5 + \frac{R_2C}{R_1+R_2}\right]e^{+at} - \frac{R_2C}{R_1+R_2}$$

10 points: If the answer was wrong, partial credits were awarded based on the formula used and correctness of approach. The distribution of points between parts a,b and c varied because of the many ways that students chose to show their work. As long as all the required work was shown in one of the parts, points were awarded.
Wrong KCL/KVL relations or branch equations were not given credit.

5. The second-order RL-circuit shown below has two resistors and two inductors.



a. Derive the differential equations using the inductor currents $i_{L1}(t)$ and $i_{L2}(t)$ as state variables and put in the matrix form (state equations).

$$L_1\frac{di_{L1}}{dt} + (R_1+R_2)i_{L1} + R_2 i_{L2} = v_s \Rightarrow \frac{di_{L1}}{dt} = -\left(\frac{R_1+R_2}{L_1}\right)i_{L1} - \left(\frac{R_2}{L_1}\right)i_{L2} + \frac{1}{L_1}v_s$$

$$\dot{x}_1 = -\left(\frac{R_1+R_2}{L_1}\right)x_1 - \left(\frac{R_2}{L_1}\right)x_2 + \frac{1}{L_1}v_s$$

$$L_2\frac{di_{L2}}{dt} + R_2 i_{L1} + R_2 i_{L2} = 0 \Rightarrow \frac{di_{L2}}{dt} = -\frac{R_2}{L_2}i_{L1} - \frac{R_2}{L_2}i_{L2}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-\frac{(R_1+R_2)}{L_1} & -\frac{R_2}{L_1}\\ -\frac{R_2}{L_2} & -\frac{R_2}{L_2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{1}{L_1}\\ 0\end{bmatrix}v_s \qquad \dot{x}_2 = -\left(\frac{R_2}{L_2}\right)x_1 - \left(\frac{R_2}{L_2}\right)x_2$$

b. Derive the characteristic polynomial for this circuit and find the roots of the equation in terms of R_1, R_2, L_1 and L_2.

$$\det\left[sI_2 - A\right] = \det\begin{bmatrix}s+\left(\frac{R_1+R_2}{L_1}\right) & +\frac{R_2}{L_1}\\ +\frac{R_2}{L_2} & \left(s+\frac{R_2}{L_2}\right)\end{bmatrix} = 0 \Rightarrow \text{Charactrist Eq}$$

$$\left[s+\left(\frac{R_1+R_2}{L_1}\right)\right]\left(s+\frac{R_2}{L_2}\right) - \frac{R_2^2}{L_1L_2} = 0 \Rightarrow s^2 + \left[\left(\frac{R_1+R_2}{L_1}\right) + \frac{R_2}{L_2}\right]s + \left(\frac{R_1+R_2}{L_1}\right)\frac{R_2}{L_2} - \frac{R_2^2}{L_1L_2} = 0$$

$$s^2 + \cancel{\frac{(R_1+2R_2)}{L}}s + \cancel{\frac{R_1R_2+R_2^2}{L^2}} - \cancel{\frac{R_2^2}{L_1L_2}} = 0 \Rightarrow \cancel{s^2 + \left(\frac{R_1+2R_2}{L}\right)s + \frac{R_1R_2}{L} = 0}$$

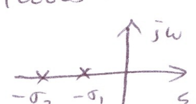
$$s^2 + \left[\frac{(R_1+R_2)L_2 + R_2L_1}{L_1L_2}\right]s + \frac{R_1R_2}{L_1L_2} = 0$$

c. Let R_1=1ohm, L_1=L_2=1H, find the value of R2 such that the circuit is critically damped. Can this circuit be underdamped? Why? Explain your answer.

$$\omega_n^2 = \frac{R_1R_2}{L_1L_2} \Rightarrow \omega_n = \sqrt{\frac{R_1R_2}{L_1L_2}};\quad 2\zeta\omega_n = \frac{(R_1+R_2)L_2+R_2L_1}{L_1L_2};\quad \zeta = \left[\frac{(R_1+R_2)L_2+R_2L_1}{2L_1L_2}\right]\frac{\sqrt{L_1L_2}}{\sqrt{R_1R_2}}$$

$$\cancel{2\zeta\omega_n = \left(\frac{R_1+2R_2}{L}\right)} \Rightarrow \cancel{\zeta = \frac{1}{2}\left(\frac{R_1+2R_2}{L}\right)\sqrt{\frac{L}{R_1R_2}}} = \cancel{\frac{1}{2}\left(\frac{R_1+2R_2}{R_1R_2\sqrt{L}}\right)}$$

Roots must be real, $s_1 = -\sigma_1$ and $s_2 = -\sigma_2$ in general



$\sigma_1 > 0, \sigma_2 > 0$

overdamped

$$\zeta = 1 \text{ (Critically damped)} \Rightarrow \left[\frac{(R_1+R_2)L_2+R_2L_1}{2\sqrt{L_1L_2}}\right]\sqrt{R_1R_2} = 1$$

$$\cancel{\frac{1}{2}\left(\frac{R_1+2R_2}{R_1R_2\sqrt{L}}\right) = 1} \Rightarrow \cancel{(R_1+2R_2) = 2R_1R_2\sqrt{L}} \quad \text{For } R_1=4$$

Let $R_1 = 1\Omega$, $L_1 = L_2 = 1H$

$$\frac{1+R_2+R_2}{2} = 1 \qquad 1+2R_2 = 2 \qquad 2R_2 = 1 \qquad \boxed{R_2 = 0.5\Omega}$$

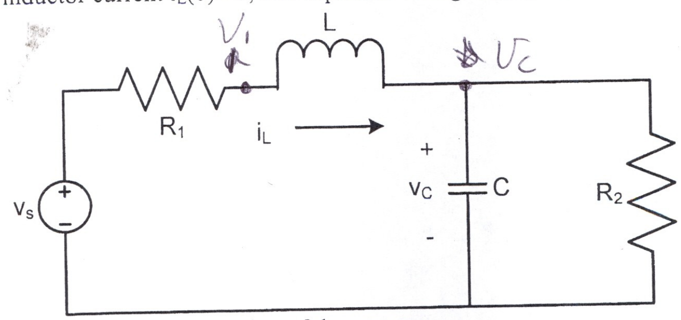
$R_1 > 0$, $R_2 > 0$
$L_1 > 0$, $L_2 > 0$

This RL-CRT CAN NOT be underdampd

10 points: If the answer was wrong, partial credits were awarded based on the formula used and correctness of approach. The distribution of points between parts a,b and c varied because of the many ways that students chose to show their work. As long as all the required work was shown in one of the parts, points were awarded.

Points were awarded to equations in part (a) as long as they were consistent with the circuit given.

6. The RLC-circuit shown below models a power distribution network with R_1, L connecting voltage source $v_s(t)$ to a load of capacitor C and resistor R_2. The voltage source $v_s(t)= u(t)$ is a step input, i.e. $u(t)= 0$ for $t< 0$, $u(t)= 1$ for $t>= 0$. We have the initial inductor current $i_L(0)= 0$, and capacitor voltage $v_C(0)= 0$.



a. Derive the state equation of the system.

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & C & 0 \\ \hline 0 & 0 & L \end{array}\right]\begin{bmatrix} \dot{v}_1 \\ \dot{v}_c \\ \dot{i}_L \end{bmatrix} = -\left[\begin{array}{cc|c} g_1 & 0 & 1 \\ 0 & g_2 & -1 \\ \hline -1 & 1 & 0 \end{array}\right]\begin{bmatrix} v_1 \\ v_c \\ i_L \end{bmatrix} + \begin{bmatrix} g_1 \\ 0 \\ 0 \end{bmatrix} v_s$$

b. Derive the inductor voltage $v_L(t)$ and capacitor current $i_C(t)$ at time $t=0^+$.

$$v_L(0^+) = L\left.\frac{di_L}{dt}\right|_{t=0^+} = V_s$$

$$i_C(0^+) = C\left.\frac{dv_c}{dt}\right|_{t=0^+} = -g_2 v_c + i_L\Big|_{t=0^+} = 0$$

c. Derive the steady state inductor current $i_L(t)$, and capacitor voltage $v_C(t)$ when time t approaches infinity.

$$i_L(t\to\infty) = \frac{V_s}{R_1+R_2}$$

$$v_C(t\to\infty) = \frac{R_2}{R_1+R_2}V_s$$

10 points:
4 points for part a: Correct equations consistent with the given circuit were awarded credits.
3 points for part b and 3 points for part c