1 Random Variables

A random variable represents a measurement that can be repeated, but each time comes out differently. We *sample* the random variable.

1. \( X : \Omega \rightarrow R \)
2. \( \omega \in \Omega \) is an outcome
3. \( A \subseteq \Omega \) is an event

Probabilities are assigned to events.

1. \( P(\Omega) = 1 \)
2. \( 1 \geq P(A) \geq 0 \)
3. if \( A \) and \( B \) are disjoint, then \( P(A \cup B) = P(A) + P(B) \)

2 CDFs

Definition. \( F_X(x) := P(X \leq x) \) Corresponds to a non-decreasing function going from \( -\infty \) at \( -\infty \) to \( 1 \) at \( +\infty \).

3 Distributions

Two flavors of distributions:

1. Discrete distributions: defined by probability on single outcomes
2. Continuous distributions: defined by a probability density function

\[
p(x) \geq 0 \text{ s.t. } \int_{-\infty}^{\infty} p(x)dx = 1
\]

Example of a mixture of discrete and continuous distributions:
1. Speeds of randomly sampled cars.

2. The wavelength of photons coming from a bright star:
   - continuous: hot body emission (lightbulb)
   - discrete: photons from specific energy releases (fluorescent bulb).

Jumps in the CDF correspond to elements with non-zero probability. Slopes correspond to densities.

4 Independence

1. Independent events $P(A \cap B) = P(A)P(B)$

2. Independent RV: $\forall E_1, E_2 \subset R$

   $$P(X \in E_1 \text{ and } Y \in E_2) = P(x \in E_1)P(y \in E_2)$$

Special cases:

- Independent discrete RV:

  $$\forall x, y \ P(X = x \text{ and } Y = y) = P(X = x)P(Y = y)$$

  [binary is a special case in which just one value pair needs to be checked]

- Independent contin. RV:

  $\forall x_1 < x_2, \ y_1 \leq y_2$ :

  $$P(x_1 \leq X \leq x_2 \text{ and } y_1 \leq Y \leq y_2) = P(x_1 \leq X \leq x_2)P(y_1 \leq Y \leq y_2)$$

5 Expected Value

$E(X) := \sum_x xp(x) = \int xdP(x)$ (Last integral is a Lebesgue integral, which is more powerful than the Riemann Integral)

Properties of expectation

- linearity: $E(aX + bY) = aE(X) + bE(Y)$
- $E(a) = a$
- $\mu = E(X)$ corresponds to the center of mass
- If $X$ and $Y$ are independent $E(XY) = E(X)E(Y)$
6 Variance

- \( Var(X) = E((X - \mu)^2) = E(X^2 - 2\mu X + \mu^2) = E(X^2) - E(X)^2 \)
- \( Var(2X) = E(4X^2) - E(2X)^2 = 4Var(X) \)

Therefore we look at std-dev = sigma(X) = sqrt(Var(X)) as a measure of the spread of the distribution.

Assume \( X_1, X_2, ..., X_n \) are all independent and consider the variance of \( \sum_{i=1}^{n} X_i \)

\[
E\left(\left(\sum_{i=1}^{n} X_i\right)^2\right) = nE(X_i^2) = nVar(X)
\]

Therefore std-dev(\( \sum_{i=1}^{n} X_i \)) scales like sqrt(N) .... remember this!!!