Post Correspondence Problem

Given two sets $A$ and $B$ of numbered strings $A_i, B_i$:

<table>
<thead>
<tr>
<th>$i$</th>
<th>$A_i$</th>
<th>$B_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>00</td>
<td>000</td>
</tr>
<tr>
<td>2</td>
<td>110</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>000</td>
<td>11</td>
</tr>
</tbody>
</table>

Is there a way to pick a sequence of numbers ($i$’s), such that:

- If you concatenate the $A$s using that sequence
- And concatenate the $B$s using that sequence
- The two concatenations are equal

Example:
Example

<table>
<thead>
<tr>
<th>i</th>
<th>Ai</th>
<th>Bi</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>00</td>
<td>010</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>1</td>
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<td>0110</td>
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Post Correspondence Problem

The Post Correspondence Problem
- Given sequence of Ai and sequence of Bi, is there a solution?
  - Undecidable problem!
  - Technical reduction that simulates the computation of a TM
Idea of proving PCP undecidable

Assume:
- TM that never moves left from LHS and never writes a blank
- Modification of PCP (MPCP) that requires solution starts with first row

- String to be generated:
  - #c_1#c_2#...#c_n
  - Each c_i a TM configuration
- A generates computation
- B generates computation, one i ahead
- A starts with:
  - #
- B starts with:
  - #qw#
- Now, each match either:
  - copies a symbol from c_i to c_{i+1} (from A to B)
  - handles a transition from c_i to c_{i+1}
  - ends the configuration
- At the end, must deal with accepting configuration

Example

q_0: On leftmost non-blank
q_1: Find first y
   x->!,R
   $->,R
   x->x,R
   y->,L
q_3: verify no x's y's
   x->x,R
   y->y,R
   ->Z,L
accept

q_2: move left to first blank
   x->x,L
   y->y,L
   $->L
   $->L
Reducing PCP to $\cap_{\text{CFG}}$

$\cap_{\text{CFG}} = \{(G_1, G_2) | G_1 \text{ and } G_2 \text{ are CFGs and } L(G_1) \cap L(G_2) = \emptyset\}$

- Reducing PCP to $\cap_{\text{CFG}}$ (PCP $\leq$ $\cap_{\text{CFG}}$)
- Make A into one grammar, B into another

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- $A \rightarrow 00A1 | 110A2 | 000A3 |
  00#1 | 110#2 | 000#3
- $B \rightarrow 000B1 | 0B2 | 11B3 |
  000#1 | 0#2 | 11#3

Reducing PCP to AMBIG$_{\text{CFG}}$

AMBIG$_{\text{CFG}} = \{G | G \text{ is an ambiguous CFG}\}$

- Reducing PCP to AMBIG$_{\text{CFG}}$ (PCP $\leq$ AMBIG$_{\text{CFG}}$)
- Make a grammar

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- $A \rightarrow 00A1 | 110A2 | 000A3 |
  00#1 | 110#2 | 000#3
- $B \rightarrow 000B1 | 0B2 | 11B3 |
  000#1 | 0#2 | 11#3
- $S \rightarrow A | B$
Reducing $\cap_{\text{CFG}}$ to $\text{ALL}_{\text{CFG}}$

$\text{ALL}_{\text{CFG}} = \{G | \text{G is a CFG and } L(G) = \Sigma^*\}$
- Create $G' = (G_1 \cap G_2)^c$
- Check whether $G' \in \text{ALL}_{\text{CFG}}$ If yes, $G_1$ and $G_2$ don’t intersect. If no, they do intersect.
- Is this valid? Is $G_1 \cap G_2$ a CFG? Is the complement of a CFL a CFL?
  - No
- However, look at $\text{PCP} \leq_{\text{CFG}} \leq_{\text{ALL}_{\text{CFG}}}$
  - The grammars used in the reduction of $\text{PCP}$ were deterministic!
  
  - So, actually, the reduction is:

  - $G' = G_1^c \cup G_2^c = (G_1 \cap G_2)^c$
  - $G^c$ is DCFG
  - Union is CFG

Reducing $\text{ALL}_{\text{CFG}}$ to $\text{EQ}_{\text{CFG/REG}}$

$\text{EQ}_{\text{CFG/REG}} = \{(G, R) | \text{G is a CFG, R is an FSA, } L(G) = L(R)\}$
- How to solve $\text{ALL}(G)$:
  - Check $\text{EQ}(G, \Sigma^*)$
Reducing $\text{EQ}_{\text{CFG/REG}}$ to $\text{EQ}_{\text{CFG}}$

$\text{EQ}_{\text{CFG}} = \{(G_1, G_2) | G_1 \text{ and } G_2 \text{ are CFGs and } L(G_1) = L(G_2)\}$

Reductions