CSE 105: Subset Proof Example

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Prove the following two languages are equal:

1. \( L_1 \) is the set of strings that have an equal number of zeros and ones, and all the zeros come before all the ones. Formally, we write it as:
   \[
   L_1 = \{ w \mid w = 0^N1^N, N = 0, 1, \ldots \}.
   \]

2. \( L_2 \) is the set of strings defined by applying the following list of recursive rules:
   \begin{enumerate}
   
   \item \( \epsilon \) is in \( L_2 \)
   \item If \( x \) is in \( L_2 \), then so is \( 0x1 \).
   \end{enumerate}

PROOF:

We must show two parts:

1. \( L_1 \subseteq L_2 \), that is every \( x \) in \( L_1 \) is also in \( L_2 \).

2. \( L_2 \subseteq L_1 \), that is every \( x \) in \( L_2 \) is also in \( L_1 \).

For the first part, we prove it by induction of the length on strings \( x \in L_1 \). For the base case, \( \epsilon \), the empty string, is in \( L_1 \) and is of length 0 and it is also in \( L_2 \) by rule 1 of the recursive process. Before we do the inductive step, notice that any string in \( L_1 \) is of even length since there are equal number of 0s and 1s making an even length. For the inductive step, let \( x \) by any string in \( L_1 \) of length 2\( k \) and assume that it is in \( L_2 \). We want to show that all strings of length 2\( k + 2 \) that are in \( L_1 \) are also in \( L_2 \). Notice that the ONLY string in \( L_1 \) of length 2\( k + 2 \) is \( 0^k1^k1 \). Since by assumption \( x = 0^k1^k \) is in \( L_2 \), we can apply rule 2) of the recursive procedure and create \( x' = 00^k1^k1 = 0^{k+1}1^{k+1} \). So we’ve showed that all strings in \( L_1 \) are also in \( L_2 \).

For the second part, we prove it by induction on the number of applications of the recursive rules used. For the base case, the empty string is in both languages (1 application of the recursive rules). For the inductive step, first assume that we’ve applied the recursive rules \( k \) times and the resulting string, \( x \), is in \( L_1 \) (i.e. \( x = 0^m1^m \) for some value of \( m \)). We show that if you apply a \( (k + 1) \)st application of the recursive rules, the resulting string \( x' \) is also in \( L_1 \). \( x' = 0(0^m1^m)1 = 0^{m+1}1^{m+1} \) which is in \( L_1 \) by definition. Therefore, all strings in \( L_2 \) are in \( L_1 \).