CSE 20: Week 7 Homework Solutions

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November 8, 2005

SF: Solutions to 2.15, 2.16, 2.17, 2.18 are found in the back of your textbook.

SF 2.17(a): False, suppose $X = \{a, b\}$ and $Y = \{c\}$ and $f(a) = f(b) = c$. If $A = \{a\}$, $f(A) = \{c\}$, and $f^{-1}(f(A)) = \{a, b\}$. Note that $\{a, b\} \neq \{a\}$.

IS: Solutions to 1.7, 1.16, 1.4 are found in the back of your textbook.

IS 1.4 (using the perturbation method): Let

$$S_n = \sum_{k=1}^{n} k^3$$

Then $S_{n+1}$ can be written as

$$S_n + (n + 1)^3 = 1^3 + \sum_{k=2}^{n+1} k^3$$

$$= 1 + \sum_{k=1}^{n} (k + 1)^3$$

$$= 1 + \sum_{k=1}^{n} (k^3 + 3k^2 + 3k + 1)$$

$$= 1 + S_n + \sum_{k=1}^{n} 3k^2 + \sum_{k=1}^{n} 3k + \sum_{k=1}^{n} 1$$

The $S_n$’s cancel out, leaving us with:

$$3 \sum_{k=1}^{n} k^2 = (n + 1)^3 - 1 - 3 \sum_{k=1}^{n} k - n$$

$$= (n + 1)^3 - 3(n)(n + 1) - (n + 1)$$

$$= (n + 1)((n + 1)^2 - 3n/2 - 1)$$

$$= (n + 1)n(n + 1/2)$$
Then we have

\[ \sum_{k=1}^{n} k^2 = \frac{(n)(n+1)(2n+1)}{6} \]

Pumpkins: The problem with the proof lies in using the incorrect base case. If we could prove a base case for sets of size 2, the proof would hold. That is, if all pumpkins are pair-wise equal, then all pumpkins are equal. However, if we apply the same argument we used to get from sets of size \( n - 1 \) to \( n \) on sets of size 1, we find a problem. Since the intersection of two sets of size 1 is empty, we cannot use transitivity to get sets of size 2. The proof by induction depends on using sets of size 2 to get to sets of size 3, and so on. This is why the proof breaks down.

Precondition and Loop Invariant: **Precondition:** \( i \in \mathbb{N}, j \in \mathbb{R} \). That is, \( i \) is a nonnegative integer and \( j \) is a real number. If you try running the algorithm with \( i \in \mathbb{R} - \mathbb{N} \), you will find that the loop will never terminate. For example, see what happens when \( i \) is -1, or 2.1.

**Loop Invariant:** \( 0 \leq 12 \leq i \) and \( \text{total} = (i-12) \times j \). A good way to discover the loop invariant is to try an example. In class we set \( i = 3 \) and \( j = 2.1 \). The values for total and 12 will be \( \{0, 3, (2.1, 2), (4.2, 1), (6.3, 0)\} \).

Inductions with Predicates: The following predicates are true:

- \( P(2) \)
- \( P(n) \rightarrow P(n-1) \) for \( n \geq 1 \)
- \( (P(2) \land P(n)) \rightarrow P(2n) \)

Prove that \( P(i) \) is true for all non-negative integers \( i \).

**Proof by induction:**

Base case (\( P(1) \) is true): \( P(2) \rightarrow P(1) \), since \( P(2) \) and \( P(n) \rightarrow P(n-1) \).

Base case (\( P(0) \) is true): \( P(1) \rightarrow P(0) \), since \( P(1) \) and \( P(n) \rightarrow P(n-1) \).

Inductive Hypothesis: \( P(n) \) is true.

We will show that \( P(n+1) \) is true.

- \( P(n) \) is true by the inductive hypothesis
- \( P(2n) \) is true, since \( (P(2) \land P(n)) \rightarrow P(2n) \)
- \( P(2n - 1) \) is true, since \( P(2n) \rightarrow P(2n-1) \)
- By the same reasoning, \( P(2n), P(2n-1), \ldots, P(n+1), P(n), \ldots, P(1) \) are all true.
- Therefore \( P(n+1) \) is true.

We have shown, using a proof by induction, that \( P(i) \) is true for all non-negative integers \( i \).