Predicate Logic

Solutions to LO \{2.6, 2.18, 2.19, 2.20, Review 7, and 9\} are in the book.

Notes:

- 2.19 - The counter-example for (b), and (c) are slightly reversed. Let \( D = \mathbb{Z} \), let \( P(x) \) be “\( x \) is even” and let \( Q(x) \) be “\( x \) is odd”. For (b), the second statement \((\exists x \in \mathbb{Z}, "x \) is even") \( \land (\exists x \in \mathbb{Z}, "x \) is odd") is true, but the first statement \( \exists x \in \mathbb{Z}, "x \) is odd" \( \land "x \) is even" is false.

- 2.20 - In more detail:

Let \( n = ab \), where \( a, b > 1 \) since \( n \) is composite. A useful equation is:

\[
\sum_{i=0}^{n-1} r^i = \frac{1 - r^n}{1 - r}
\]

Applying this to \( r = 2 \) we get

\[
2^0 + 2^1 + \ldots + 2^{n-1} = \sum_{i=0}^{n-1} 2^i
= \frac{1 - 2^n}{1 - 2}
= \frac{1 - 2^n}{-1}
= 2^n - 1
\]

Since we want to show \( 2^n - 1 \) is composite, the summation could come in handy. If we can factor \( 2^n - 1 = cd \), then it is composite. Perhaps selectively summing some of the terms will be equivalent to factoring. Since \( n = ab \) is already factored, maybe sum terms with an exponent a multiple of \( a \) or \( b \):

\[
2^{0a} + 2^{1a} + \ldots + 2^{k a}
\]
We don’t want can $k \cdot a$ to exceed $n - 1$ so:

\[
k \cdot a \leq n - 1
\]

\[
k \leq \frac{n - 1}{a}
\]

\[
k \leq b - \frac{1}{a}
\]

\[
k \leq b - 1
\]

The last step is to ensure $k$ is an integer. Back to the summation:

\[
2^{0}a + 2^{1}a + \ldots + 2^{(b-1)}a = \sum_{i=0}^{b-1} (2^{a})^{i}
\]

\[
= \frac{1 - (2^{a})^{b}}{1 - 2^{a}}
\]

\[
= \frac{1 - 2^{n}}{1 - 2^{a}}
\]

\[
= \frac{2^{n} - 1}{2^{a} - 1}
\]

Let $c = \frac{2^{n} - 1}{2^{a} - 1}$ and $d = 2^{a} - 1$. Then

\[
c \cdot d = \frac{2^{n} - 1}{2^{a} - 1} \cdot (2^{a} - 1) = 2^{n} - 1
\]

and we have factored $2^{n} - 1$. This is valid because $c$ and $d$ are both integers $> 1$. $c$ represents the sum $2^{0}a + 2^{1}a + \ldots + 2^{(b-1)}a$ of integers. This makes $c$ an integer $> 1$. Also, since $a$ is an integer $> 1$, $d = 2^{a} - 1$ is an integer $> 1$.

- Answers to review questions 7 and 9 are after all the questions.

## Number Theory

Solutions to NT \{1.3, 1.8, 1.11, 1.17\} are in the book.

Notes:

- A good way to prove “if and only if” statements is to prove the “if” part separately from the “only if” part. Let’s look at 1.3(a) for example.

  Prove: The product of two integers is even if and only if at least one of them is even.
⇒ (the “if” part): Assume the product of two integers $x \cdot y$ is even (this is actually a given for the “if” part). So $x \cdot y = 0 \mod 2$. For the sake of contradiction, assume both numbers are odd. That is, $x = 1 \mod 2$ and $y = 1 \mod 2$. Then $x \cdot y = 1 \mod 2$, a contradiction. Therefore, at least one of $x, y$ must be even.

⇐ (the “only if” part): You no longer assume that the product $xy$ is even. That is what you must show now. However, you are now given that at least one of the integers is even. Without loss of generality say $x$ is even (otherwise, relabel the two). So $x = 0 \mod 2$. Well, 0 times anything is 0 $\mod 2$. Therefore, $xy = 0 \mod 2$, meaning the product is even.