Predicates

Grammatical origin

Alternate definitions of a predicate:
- A sentence with a finite number of variables. Replacing variables with specific values yields a statement. The domain is the set of all possible values to substitute.
- A function whose codomain is (true/false) statements.

Example:
- \( x^2 > 1 \)
  - Domain?
  - Example statements?
- “Boy x likes girl y” (alternate syntax:)
  - Domain?
  - Example statements?

Truth Set of a Predicate, \( P(x) \)
The set of all elements in the domain that make \( P(x) \) true

Denoted:
- \( \{ x \in D : P(x) \} \)

Example:
- What is the truth set of \( \text{“x is a factor of 8”} \)?
  - If \( D = \{0, 1, 2\} \)?
  - If \( D = \{0, 1, 2, 3, \ldots\} (\mathbb{N}) \)?
  - If \( D = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} (\mathbb{Z}) \)?

Quantifiers

Universal
- \( \forall x \in D, S(x) \)

Existential
- \( \exists x \in D : S(x) \)

Common to drop \( \in D \) if it is clear from context
Examples

Let $L(x, y)$ be the predicate “Boy $x$ likes girl $y$”

- $\exists y: \exists x: L(x, y)$
- $\exists y: \forall x: L(x, y)$
- $\forall x \exists y L(x, y)$
- $\forall y L("Jose", y)$

Negation and Quantifiers

$\neg (\exists x: P(x)) \leftrightarrow \forall x \neg P(x)$

$\neg (\forall x P(x)) \leftrightarrow \exists x: \neg P(x)$

Examples

$L(x, y) = "x$ likes $y" \quad \text{Domain: all people in my son's 1st grade class}$

- Everyone likes Jill
- Nobody likes Jill
- Not everybody likes Jill
- There exists a person that nobody likes
- Nobody dislikes everybody
- Everybody likes him/herself
- $\forall x, \forall y, \forall z, L(x, y) \land L(y, z) \rightarrow L(x, z)$
- $\forall x, \forall y, L(x, y) \rightarrow L(y, x)$

Converting to Predicates

If a number is a natural number, it is an integer

The square of any real number greater than 2 is greater than 4
Quantifiers and Adversaries

Statements involving quantifiers can be viewed as a game
- Two players
  - You, trying to prove the statement is true
  - An adversary, trying to prove the statement is false
- Work from left to right through the quantifiers
  - ∀y: the adversary selects an y (think of the adversary as choosing the worst possible y).
  - ∃x: you select an x that will satisfy the statement. That x remains bound the rest of the game

Example:
- ∀x ∈ Z, ∃y ∈ Z: 2x = y
- ∃x ∈ Z: ∀y ∈ Z, xy = 0
- ∀y ∈ Z, ∃x ∈ Q: xy = 1

Limit

The limit of a sequence a is L means that a gets arbitrarily close to L

\[ \lim_{n \to \infty} a_n = L \]

\[ \forall \varepsilon \in R^+, \exists n_0 \in N : \forall n \in N, n \geq n_0 \rightarrow L - \varepsilon \leq a_n \leq L + \varepsilon \]

Order of Growth of a Function

We say that T(n) has an asymptotic upper bound of f(n) if:

\[ \exists n_0 \in N : \exists c \in R^+ : \forall n \in Z, n \geq n_0 \rightarrow 0 \leq T(n) \leq cf(n) \]

Proving Quantified Statements

Prove universal statement: ∀x ∈ D, P(x) → Q(x)
- Exhaustive enumeration
- Generalizing from the generic particular
  - “Suppose x is in D and P(x)”
  - “…”
  - Therefore Q(x)
- Example: The difference of two odd numbers is even
Proving Quantified Statements

Prove existential statement: \( \exists x \in D: P(x) \)
- Constructive proof
  - Display an \( x \)
  - Give a set of directions for finding \( x \)
- Nonconstructive proof
  - Proof by contradiction (assume non-existence and show a contradiction)
  - Show \( x \) must exist
- Example:
  - In a group of 367 people, at least two share a birthday

Disproving Quantified Statements

Disprove universal statement: \( \forall x \in D, P(x) \rightarrow Q(x) \)
- Counterexample
  - Show an \( x \) in \( D \) where \( P(x) \), but not \( Q(x) \)
- Example: All primes are of the form \( 2^n - 1 \)

Disprove existential statement: \( \exists x \in D: P(x) \)
- Equivalent to:
  - Prove
  - Or, alternatively,
    - Therefore, best bet is generalizing from the generic particular.
- Example: There exists a prime which can be written as the square of an integer > 1