Logic

Propositional Logic
- Deals with individual statements that have true/false values
  - no in-between
- Corresponds 1-1 with Boolean Functions and Combinational Circuits

(First-order) Predicate Logic
- Can actually talk about groups: For all, There exists,
- Covered next lecture

Propositional Logic Formulas

Well-formed formulas (wff)
- Any lower-case letter is a wff (atomic formula)
  - p
  - q
  - r
  - ...
- If \( \Phi \) is a wff, then \( \neg \Phi \) is a wff
- If \( \Phi \) and \( \Pi \) are wffs, then:
  - \( (\Phi \vee \Pi) \) is a wff
  - \( (\Phi \wedge \Pi) \) is a wff
  - \( (\Phi \rightarrow \Pi) \) is a wff
  - \( (\Phi \leftrightarrow \Pi) \) is a wff

As a shortcut, we sometimes omit unneeded parentheses

Simplifying a Boolean Function

Example
- \( \neg (p \wedge q) \vee (p \vee q) \)
Formula

An atomic formula (p) represents some statement that has a truth value:
- Either p is true, or it’s negation is true. 
  - tertium non datur
- It’s either true or false (bivalent)

Example English propositions:
- It will rain tomorrow
- The sky is blue
- The sky is fluorescent orange
- The program will finish in less than 3 minutes

The truth value of a complex formula can be determined from the truth values of the atomic formulas within.

Equivalent Formulas

Formulas that differ syntactically are said to have different forms:
- Formulas with different forms are equivalent if their truth tables are the same.
- Example:

Implication

Implication (also called if-then):
- p → q
- For the formula to be true, if p is true, then q can’t be false.
- If p is false, the formula is true
- Example:
  - If it is raining tomorrow, the driveway will get wet
### Implication

**Common English forms**

- *If* \( p \) *then* \( q \)
  - If it is raining, then the driveway will get wet

- \( p \) *only if* \( q \)
  - The team will play in the playoffs *only if* the team reaches the semifinals

- \( p \) *if and only if* \( q \)
  - The team will be the champion *if and only if* the team wins the championship game

- \( p \) *is necessary for* \( q \)
  - A necessary condition for \( q \) is \( p \)
    - Reaching the semifinals is necessary for playing in the playoffs

### Relatives of Implication

**Given** \( p \rightarrow q \)

- Inverse: \( \neg p \rightarrow \neg q \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>( p \rightarrow q )</th>
<th>( \neg p \rightarrow \neg q )</th>
</tr>
</thead>
<tbody>
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- Converse: \( q \rightarrow p \)

### Anything can be Proved from a Contradiction

**If we can establish:**

- A contradiction implies anything because:

  - \( \neg \rightarrow q \)
  - \( \neg \text{contradiction} \)
  - Therefore, \( q \)
Proof by Contradiction

To prove \( q \) by contradiction

- Assume \( \neg q \)
- Show this leads to a contradiction

Example
- Prove that there is no largest integer \( p \)
- Assume, to the contrary that \( \neg p \)
- Then, \( \text{formula that follows from } p \)
- Now, \( p \land \neg p \)

Argument forms

Classical logic uses syllogisms

- Two premises
- One conclusion

- \textit{Modus ponens}
  - If it rained, the driveway got wet
  - It rained
  - Therefore, the driveway got wet

- \textit{Modus tollens}
  - If it rained, the driveway got wet
  - The driveway didn’t get wet
  - Therefore, it didn’t rain

Argument Forms

\begin{itemize}
  \item Generalization
    \begin{itemize}
      \item \( p \)
      \item Therefore, \( p \lor q \)
    \end{itemize}
  \item Specialization
    \begin{itemize}
      \item \( p \land q \)
      \item Therefore, \( p \)
    \end{itemize}
  \item Conjunction
    \begin{itemize}
      \item \( p \)
      \item \( q \)
      \item Therefore, \( p \land q \)
    \end{itemize}
  \item Elimination
    \begin{itemize}
      \item \( p \lor q \)
      \item \( \neg q \)
      \item Therefore, \( p \)
    \end{itemize}
  \item Transitivity
    \begin{itemize}
      \item \( p \rightarrow q \)
      \item \( q \rightarrow r \)
      \item Therefore, \( p \rightarrow r \)
    \end{itemize}
\end{itemize}
Algebraic Rules for Propositional Formulas

Equivalences between propositional formulas (similar to algebraic equivalences):

- Associative
- Distributive
- Idempotent
- Double negation
- DeMorgan’s
- Commutative
- Absorption
- Bound
- Negation