Infinite Series

Our approach will not be based on Bender. You are not responsible for IS section 3

- But, you are responsible for this lecture!

Paradoxes with Infinite Sums

\[ S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \]

\[ S = \text{____________________} \]

\[ T = 1 + 2 + 4 + 8 + 16 + \ldots \]

\[ T = \text{____________________} \]

Bound of \( \sum_{k \in K} a_k \)

Assume each \( a_k \) is non-negative

K may be infinite

If there exists a constant A such that, for all finite subsets \( F \subset K \):

\[ \sum_{k \in F} a_k \leq A \]

- Then, we say \( \sum_{k \in K} a_k \) converges to the least such A, \( A_{\text{min}} \)
- If there is no such A, then we say the sum diverges to infinity.

If \( K \) is the set of non-negative integers:

\[ \sum_{k \in K} a_k = \lim_{n \to \infty} \sum_{k=0}^{n} a_k \]
Examples

\[ \sum_{k \geq 0} x^k = \lim_{n \to \infty} \frac{1 - x^{n+1}}{1 - x} \]

- if \(0 \leq x < 1\):
- if \(x \geq 1\):

\[ S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \]

\[ T = 1 + 2 + 4 + 8 + 16 + \ldots = \]

Negative Terms in Infinite Series

Three different answers

\[ \sum_{k \geq 0} (-1)^k \]

- \(1 + -1 + 1 + -1 + \ldots \)
- \(1 + -1 + 1 + -1 + \ldots \)
- Use formula for Geometric series

Negative Terms in Infinite Series

Two different answers

\[ \sum_{k \geq 0} \frac{1}{k+1} + \sum_{k < 0} \frac{1}{k-1} \]

- \(\ldots + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \)
- Sum of \(n\) innermost parentheses =

\[ \ldots + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \]
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\[ \ldots + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \]
- Sum of \(n\) innermost parentheses =

Infinite Sums with Positive and Negative Terms

Break apart sums into positive terms and negative terms

- Define \(x^+\) and \(x^-_\):
  - If \(x \geq 0\), \(x^+ = x\), \(x^- = 0\)
  - If \(x < 0\), \(x^+ = 0\), \(x^- = x\)

\[ \sum_{k \in K} a_k = \sum_{k \in K} a^+_k - \sum_{k \in K} a^-_k \]

Let \(A^+ = \)

If \(A^+\) and \(A^-\) are finite
- Sum converges \textit{absolutely} to \(A^+ - A^-\)

If \(A^+\) is infinite, but \(A^-\) is finite:
- Sum diverges to infinity

If \(A^+\) is finite, but \(A^-\) is infinite:
- Sum diverges to negative infinity

If \(A^+\) and \(A^-\) are infinite
- Sum is undefined
Conditional Convergence

A sum where $\sum_{k \geq 0} a_k$ converges, but $\sum_{k \geq 0} |a_k|$ doesn’t

- Example (alternating harmonic series)
  - $-1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \ldots = \ln 2$

- Kind of wacky, because rearranging the order of terms in the series can yield different values to which the series converges
  - In fact, can rearrange the order to obtain any desired value!

- $\sum_{k \geq 0} (-1)^k \frac{1}{k+1} + \sum_{k \leq 0} \frac{1}{k-1}$

Examples

- $\sum_{k \geq 0} \frac{1}{k}$
- $\sum_{k \geq 0} \frac{1}{k+1} + \sum_{k \leq 0} \frac{1}{k-1}$

Advantages of absolute convergence

- Can continue to use distributive, commutative, and associate laws of sums

Harmonic Series

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \ldots$

- Lower and upper bounds on:
  
  $H_n = \sum_{1 \leq k \leq n} \frac{1}{k}$

- Put one term in group 1, two in group 2, four in group 3, etc.
  - $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \ldots$

- Each group has terms no bigger than first term and strictly bigger than first term of next group.
  - $\sum_{1 \leq k \leq n} \frac{1}{k} < \text{sum of each group} \leq H_n \leq \sum_{1 \leq k \leq n} \frac{1}{k}$
  - If $n$ is in group $k$, $\frac{\lfloor \log n \rfloor}{2} < H_n \leq \lfloor \log n \rfloor + 1$

  Gives bound to $H_n$ within a factor of 2

Better Bounds for Harmonic Series

Use integration of $f(x) = \frac{1}{x}$

- Upper bound:
  - $\int_1^n \frac{1}{x} \, dx = \ln n$

- $H_n > \ln n$

- $\ln n < H_n < n + 1$

- $H_n < \ln n + 1$

Bounds to $H_n$ within at most 1