Midterm 2

1. 8 pts. Given any function, \( f \), which of the following are true about the domain, codomain, image, and coimage of \( f \)? Circle all that are true.
   (a) The size of the coimage is equal to the size of the image.
   (b) The size of the codomain is equal to the size of the domain.
   (c) The image is a partition of the codomain.
   (d) The coimage is a partition of the domain.
   (e) The coimage is a subset of the domain.
   (f) The image is a subset of the codomain.
   (g) The size of the coimage is no bigger than the size of the domain.
   (h) The size of the image is no bigger than the size of the codomain.

2. 8 pts. Following are some statements about the greatest common divisor (gcd) and least common multiple (lcm) functions and integers \( m \) and \( n \). Circle all that are true.
   (a) \( \gcd(m,n) \times \text{lcm}(m,n) = m \times n \).
   (b) \( \gcd(m,n) = \gcd(m-cn,n) \) for all integers \( c \).
   (c) \( \gcd(m,n) = kn + jn \) for some integers \( k \) and \( j \).
   (d) \( \gcd(m,n) = 1 \) iff \( m \) and \( n \) are relatively prime.
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3. 10 pts. Give a partition of \{a, b, c, d, e, f, g, h\} of cardinality 3.

5. 15 pts. Prove using induction that for all integers \( n \geq 1 \):

\[
\sum_{k=1}^{n} k! = (n+1)! - 1
\]

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4. 12 pts. What is the value of the following sum and why?

\[
\sum_{i=0}^{n} \binom{n}{i}
\]

6. 15 pts. Let \( A = \{a,b,c,d\} \) and \( B = \{1,2,3\} \). For this exercise, answer each part with an actual number as well as showing your work.

[HINT: recall that the Stirling numbers are defined as \( S(n,k) = S(n-1,k-1) + kS(n-1,k) \).]

(a) How many surjective functions are there with domain \( A \) and codomain \( B \)?

(b) How many non-surjective functions are there with domain \( A \) and codomain \( B \)?

(c) How many injective functions are there with domain \( B \) and codomain \( A \)? [Note: the domain and codomain are reversed from parts a) and b.]
7. 10 pts. Here is a mistaken “proof” that for all sets $A$ and $B$, $A^c \cup B^c \subseteq (A \cup B)^c$.

Proof: Suppose $A$ and $B$ are sets, and $x \in A^c \cup B^c$. Then $x \notin A$ or $x \notin B$ by definition of complement, and so $x \notin A \cup B$ by definition of union. Thus, $x \in (A \cup B)^c$ by definition of complement, and hence $A^c \cup B^c \subseteq (A \cup B)^c$.

Circle the mistake in this proof and explain why it is wrong.

8. 12 pts.
Define $A \times B \times B$ as $\{(r,s,t) : r \in A, s,t \in B\}$. Given sets $A = \{a,b,c\}$ (with $a < b < c$), $B = \{0, 1, 2, 3, 4\}$ (with $0 < 1 < 2 < 3 < 4$), consider $C = (A \times B) \cup (A \times B \times B)$.

(a) What is the first member of $C$ in lexicographic order?

(b) What is the second member of $C$ in lexicographic order?

(c) What is the second-to-last member of $C$ in lexicographic order?

9. 15 pts. Extra Credit. Evaluate the following sum to come up with a closed form (simplify your result as much as possible). [Note: you can consider using the perturbation method, or guessing the solution and then proving it with induction.]

$$S_n = \sum_{0 \leq k \leq n} k(-1)^k$$

Monotonicity

A sequence $a_n$ is
- **strictly increasing** if each term is bigger than the previous one
- **strictly decreasing** if each term is smaller than the previous one
- **nondecreasing** (or weakly increasing) if each term is no smaller than the previous one
- **nonincreasing** (or weakly decreasing) if each term is no bigger than the previous one
- **monotonic** (or monotone) if it is either nondecreasing or nonincreasing

A sequence is **eventually [...] if some tail of the sequence is [...]**

Examples
- $1, 3, 5, \ldots$
- $x^2-8x+1$: 
Bounded and Monotone

If a sequence is bounded and eventually monotone, then it converges.
Example:
- limit of $2 + \frac{1}{n}$
- limit of $2 - \frac{1}{n}$

Sandwich theorem:
- If $a_n$ has a limit of $A$ and $c_n$ has the same limit, and each term of $b_n$ has the property that:
  - $a_n \leq b_n \leq c_n$
  - Then, the limit of $b_n$ is $A$

Example:
- limit of $2 + (-1)^n$ / $n$

Sequence Cagematch

Given two sequences $a_n$ and $b_n$, which grows faster?
- Polynomial of degree $n$ versus polynomial of degree $n-1$
- Exponential versus polynomial?
- Polynomial versus logarithmic?

Identifying a Sequence

The Online Encyclopedia of Integer Sequences

Enter part of a sequence:
- 0 1 3 6 10 15 21

Get back a list of known related sequences

ID Number: A000217 (formerly M2535 and N1002)


Sequence:
0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561, 595, 640, 686, 733, 781, 830, 881, 933, 986, 1041, 1098, 1157, 1218, 1281

Name: Triangular numbers: a(n) = C(n+1,2) = n(n+1)/2 = 0+1+2+...+n.

Comments:
- Number of edges in complete graph of order n, K_n.
- Number of legal ways to insert a pair of parentheses in a string of n letters. E.g. there are 4 ways for these letters: (a, b) (c, d), (a, (b, c)) (d), (a, b, (c, d)). The first two make the parentheses safe, the last two do not. No parentheses will be adjacent because the parentheses are adjacent. Cf. A002415.
- For n >= 1 a(n)=n(n+1)/2 is also the genus of a nonsingular curve of degree n+2 like the Fermat curve x^(n+2) + y^(n+2) = 1 - Ahmed Fares (ahmedfares(AT)my_deja.com), Feb 21 2001
- a(n) is the number of ways in which n+2 can be written as a sum of three positive integers if representations differing in the order of the terms are considered to be different. It also counts the number of positive integer solutions of the equation x + y + z = n+2. - Amarnath Murthy (amarnath.murthy(AT)yahoo.com), Apr 22 2001

For n >= 1 a(n)=(n+1)/2 is also the genus of a nonsingular curve of degree n+2 like the Fermat curve x^(n+2) + y^(n+2) = 1 - Ahmed Fares (ahmedfares(AT)my_deja.com), Feb 21 2001