Functions

Give two sets A and B, a function is a rule that tells how to find a unique \( b \in B \) for each \( a \in A \).

- It is shown as \( f : A \to B \)
- A is the \textit{domain}
- B is the \textit{range} (or \textit{codomain})

- The set of all functions from A to B is written \( B^A \).
- \( f \in B^A \to f : A \to B \)
- Given a set \( S \subseteq A \), \( f(S) = \{ f(x) : x \in S \} \)
- \( f(\emptyset) = \emptyset \)
- \( f(A) \) is called \textit{image}

Ways of showing the rule (given domain and range: \( A = \{ \text{red, green, blue} \} \), \( B = \{0..28\} \)):

- One-line notation
  - If A is ordered, give the corresponding values of B:
    - Ordering of A = (red, green, blue), \( f = (5, 8, 27) \)

- Two-line notation
  - Give ordering of A and values of B at the same time:
    - \( f = \begin{pmatrix} \text{red} & \text{green} & \text{blue} \\ 5 & 8 & 27 \end{pmatrix} \)

- Set notation: \( \{( \text{red, 5}), (\text{blue, 27}), (\text{green, 8})\} \)

Types of Functions

Surjection (onto): Every value in the range is taken on at \textbf{least} once
- \( \forall b \in B, \exists a \in A: f(a) = b \)

Injection (one-to-one): Every value in the range is taken on at \textbf{most} once.
- \( \forall a \in A, b \in B, f(a) = f(b) \rightarrow a = b \)

Bijection (one-to-one and onto): Every value in the range is taken \textbf{exactly} once.
- \( \forall b \in B, \exists! a \in A: f(a) = b \)

Example

Let
- \( S \) be the set of students attending UCSD
- \( I \) be the set of student ID numbers for those students
- \( D \) be the set of dates (MM/DD/YYYY) for the past 100 years
- \( G \) be the set of possible grade point averages between 2.0 and 3.5 (rounded to the nearest tenth)

Is the following an injection, bijection, or surjection?
- The domain is \( S \), the codomain is \( I \) and the function maps each student to his/her ID number
- The domain is \( S \), the codomain is \( D \) and the function maps each student to his or her birthday
- The domain is \( D \), the codomain is \( I \) and the function maps each date to the ID number of a student born on that date. If more than one, the lexicographically least ID number is chosen.
- The domain is \( S \), the codomain is \( G \) and the function maps each student to his or her GPA rounded to the nearest tenth.
- The domain is \( G \), the codomain is \( I \) and the function maps each GPA to the ID number of a student with that GPA. If there is more than one, the lexicographically least ID number is chosen.
Example

Bijective, injective, surjective, or neither?
- $f: \mathbb{R} \to \mathbb{R} \quad f(x) = x$
- $f: \mathbb{R}^+ \to \mathbb{R}^+ \quad f(x) = x^2$
- $f: \mathbb{R} \to \mathbb{R}^+ \quad f(x) = x^2$
- $f: \mathbb{R} \to \mathbb{R} \quad f(x) = \sqrt{x}$
- $f: \mathbb{R} \to \mathbb{R} \quad f(x) = (x-1)x(x+1)$
- $f: \mathbb{R} \to \mathbb{R} \quad f(x) = e^x$
- $f: \mathbb{R} \to \mathbb{R}^+ \quad f(x) = e^x$

Bijections

Given two sets $A$ and $B$
- $|A| = |B|$ iff there exists a bijection $f: A \to B$

Inverse Functions

If $f: A \to B$ is a bijection, then $f^{-1}$ is the inverse of $f$. $f^{-1}: B \to A$
- if $f(a) = b$, then $f^{-1}(b) = a$

Numbers of Functions

How many functions of the form $f: A \to B$?
- Equivalently, what is $|B^A|$?
  - How many choices for first element of $A$?
  - How many choices for second element of $A$?
  - ...
  - How many choices for last element of $A$?
Relations

If A and B are sets, a subset, R, of A × B is a relation from A to B.
- Given relation R, if, for all a in A, there is a unique b in B such that (a, b) in R, then R is a functional relation.
- ∀a ∈ A, ∃!b ∈ B: (a, b) ∈ R.
- (x, y) ∈ R ↔ xRy

Examples:
- A = B = [-1, 1], R={(x, y): x² + y²=1}
- A = [-1, 1] B = [0, 1], R={(x, y): x²+y²=1}
- A = B = R, R = {(x, y), y = x³}

Permutations

A permutation is a bijection from A to A.

Composition:
- If f and g are two functions where the values taken on by f are all in the domain of g,
  - equivalently, f: A → B and g: C → D and f(a) ∈ C for all a in A
- The composition of f and g (gf: A → D, or g · f: A → D) is defined as (gf)(x) = g(f(x)).

If f and g are permutations of A, then fg is a permutation of A, and so is f⁻¹.
If f is a permutation of A, we write f² instead of ff.

Cycles

Look at an element x in a permutation f of A.
- Look at the sequence of values x, f(x), f²(x), ...
- Eventually, it will repeat (how long, max?)

If p is the first value for which fp(x) = x, then the cycle (of length p) containing x is:
- (x, f(x), f²(x), ..., fp⁻¹(x))

Any permutation can be rewritten as a set of cycles:
- If f maps (1, 2, 3, 4, 5, 6, 7, 8, 9) to (2, 4, 8, 1, 5, 9, 3, 7, 6), then the cycles are:
  - (1, 2, 4), (3, 8, 7, 10), (5), (6, 9)
- Can leave off cycles of length 1 (if domain is clear):
  - (1, 2, 4), (3, 8, 7), (6, 9)
- Inverse of the function is the cycles in reverse order:
  - (4, 2, 1), (7, 8, 3), (9, 6)

Powers of Permutations

To calculate the power of a function f, calculate the power of each cycle:
- Calculate final result for each element (using i mod cycle-length)
- 1-cycles stay 1-cycles
- For example, if f = (1, 2, 4), (3, 8, 7, 10), (5), (6, 9)
  - f⁵ = (1, 2, 4), (3, 7), (8, 10), (5), (6), (9)
- If all cycle lengths divide i evenly, fi is the identity permutation.
- If f is a permutation of A, then f|A| is the identity permutation.