Proof of the Convolution Theorem

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$$f(x) * h(x) \overset{\Im}{\longleftrightarrow} F(u)H(u) \tag{1}$$

$$g(x) = \frac{1}{M}\sum_{x=0}^{M-1} f(k)h(x-k) \tag{2}$$

Perform a Fourier Transform on each side of the equation:

$$G(u) = \frac{1}{M}\frac{1}{M}\left(\sum_{x=0}^{M-1}\left(\sum_{k=0}^{M-1} f(k)h(x-k)\right)e^{-j2\pi ux/M}\right) \tag{3}$$

Factor 1 into two exponentials and subsitute them into the equation:

$$= \frac{1}{M^2}\sum_{x=0}^{M-1}\sum_{k=0}^{M-1} f(k)h(x-k)e^{-j2\pi uk/M}e^{j2\pi uk/M}e^{-j2\pi ux/M} \tag{4}$$

Note: f(x) and h(x) are both assumed to be periodic with period M:

$$= \frac{1}{M^2}\sum_{k=0}^{M-1} f(k)\left(\sum_{x=0}^{M-1} h(x-k)e^{-j2\pi u(x-k)/M}\right)e^{-j2\pi uk/M} \tag{5}$$

Since $\sum_{x=0}^{M-1} h(x)e^{-j2\pi ux/M} = H(u)\cdot M$:

$$G(u) = \frac{M}{M^2}\sum_{k=0}^{M-1} f(k)H(u)e^{-j2\pi uk/M} \tag{6}$$

$$= \frac{1}{M}F(u)H(u)\cdot M \tag{7}$$

$$= F(u)H(u) \tag{8}$$