CSE 202 - Algorithms

Quicksort vs Heapsort:

the "official" story

Next time, we'll get the "inside" story!

Proof that $O(g(n)) = O'(g(n))$

Given $g:\mathbb{N}\to\mathbb{Z}^+$, we will first show $O'(g(n)) \subseteq O(g(n))$; then we'll show that $O(g(n)) \subseteq O'(g(n))$.

Let $f \in O'(g(n))$ where $f:\mathbb{N}\to\mathbb{Z}^+$.

By the definition of O', $\exists c>0\ \forall n,\ 0 < f(n) \le c\, g(n)$.
Let $n_0 = 0$. Clearly $\forall n>n_0,\ 0 \le f(n) \le c\, g(n)$. Thus, $f \in O(g(n))$.

Now let $f \in O(g(n))$ where $f:\mathbb{N}\to\mathbb{Z}^+$.

By definition, $\exists c>0\ \exists n_0\ \forall n>n_0,\ 0 \le f(n) \le c\, g(n)$. (Equation I)
Let $c' = \max(c, f(0), f(1), \ldots, f(n_0))$. Clearly $c'>0$.
Given any n, if $n \le n_0$, then $f(n) \le c'$ (by the def. of max) and so $0 < f(n) \le c'\, g(n)$ (since $f(n)$ and $g(n)$ are both ≥ 1.)
Conversely, if $n>n_0$, then $f(n) \le c\, g(n)$ from Equation I. But $0 < f(n)$ and and $c' \ge c$, so $0 < f(n) \le c'\, g(n)$.
Thus, we've shown $\forall n,\ 0 < f(n) \le c'\, g(n)$, and so $f \in O'(g(n))$.

Q.E.D

Overview

- Quicksort and Heapsort are comparable
  - Both use only binary comparisons
  - Both sort in-place
- Heapsort:
  - worst-case complexity is $\theta(n \lg n)$
- Quicksort:
  - worst-case complexity is $\theta(n^2)$.
  - average-case complexity is $\theta(n \lg n)$
  - probabilistic analysis is also $\theta(n \lg n)$
- Yet Quicksort is often considered superior.

Priority queue

A "task" is an object with a "key" field

Key of task x is x.key or key(x) (or whatever style you like).

We'll just worry about the priorities.

A (max-) <u>priority queue</u> is a data structure that has the following operations:

- Insert(S, x) - add task x to the queue S.
- Extract-Max(S) - return the task with largest key and remove it from heap.
- Max(S) - return task with largest key (don't remove it).
- Increase-Key(S,x,k) – Increase x's key to be k

  (return error code if x.key is already > k.)

Heaps can implement priority queues

A <u>heap</u> is a binary tree:

- All levels except bottom are completely filled in.
- Bottom level is filled in from left (no holes).
- Has <u>heap property</u>: parent's key ≥ either child's

A heap can be stored in an array H:

- Root is H[1].
- Left child of H[k] is H[2k]
- Right child of H[k] is H[2k+1]

Heaps can implement priority queues

Insert(S, x) - add task x to the queue S.

- Add x as new last node.
- "Bubble up" to re-establish heap property.

Extract-Max(S) - return the task with largest key and remove it from heap.

- Pull task from top of heap (it has largest key).
- Replace it with the last node of heap.
- "Bubble down" (heapify) to re-establish heap property.

Increase-Key(S, x, k) – Increase x's key to be k.

- Report error if x.key > k
- Set x.key = k.
- "Bubble up" to re-establish heap property.

Exercise

- Pick a random permutation of {1,2,3,4,5,6}
- Insert these priorities into heap in the chosen order.
- Now do two Extract-Max's.

Heapsort

- Insert all the n data items into a heap, then extract them all.
- Insert and Extract-Max operations use at most c lg n time.

  Aside: does this follow from our theorem,

  "If T is a non-empty binary tree of height h, then T has fewer than 2^{h+1} nodes" ??

- So T(n) = time to sort n items < 2 n c lg n,

  $T(n) \in O(n \lg n)$.

Build-Heap

Builds heap from set S using O(n) operations (n=|S|).

Still, Heapsort's asymptotic complexity is O(n lg n), since you need n Extract-Max's.

Stuffs S into a binary tree, then massages it *from last parent to first* to establish heap property.

No comparisons needed for leaves.

Each node at level h-i needs at most 2i comparisons.

Comparisons bounded by:

$n/2 \times 2\times1 + n/4 \times 2\times2 + n/8 \times 2\times3 + \ldots + 1 \times 2\times \lg n$

$= 2n\,(1/2 + 2/4 + 3/8 + 4/16 + \ldots + \lg n/n)$

Summing $i/2^i$

$$
\begin{array}{rl}
1/2 + 1/4 + 1/8 + 1/16 + 1/32 + \ldots & \leq 1 \\
1/4 + 1/8 + 1/16 + 1/32 + \ldots & \leq 1/2 \\
1/8 + 1/16 + 1/32 + \ldots & \leq 1/4 \\
1/16 + 1/32 + \ldots & \leq 1/8 \\
\ldots & \ldots \\
\hline
1/2 + 2/4 + 3/8 + 4/16 + 5/32 + \ldots & \leq 2
\end{array}
$$

The two ways to build a heap

- Use Build-Heap

  $T(n) < 2n\,(1/2 + 2/4 + 3/8 + \ldots) = 2n \times 2 = 4n$

- Make repeated calls on Insert(H,x)

  In the worst case, each insertion requires "bubbling up" all the way to root.

  For half the nodes, this takes (lg n)-2 comparisons.

  So $T(n) > (n/2)(\lg n - 2)$, i.e. $T(n) \in \Omega(n \lg n)$

  (Average case may not be so bad.)

- Intuition why Build-Heap is (or might be) better:

  Most nodes in a heap are close to the leaves.

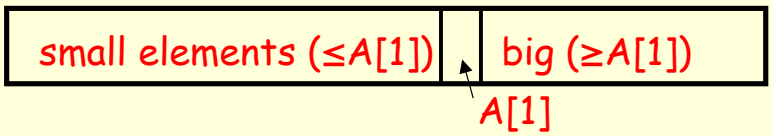
  Most nodes in a heap are far from the root.

Quicksort

Basic idea:

Split w.r.t A[1]

Arrange A as:



Recursively arrange "small" part and "big" parts.

Doesn't need extra array.

Keep pointers to current ends of small & big parts.

"Pick up" A[1] (splitter) and A[n] (current element),

leaves space in to deposit element in either part

drop current element appropriately; pick up next innermost.

Worst-case Quicksort complexity

What happens if A is already sorted?

$T(n) = T(1) + T(n-1) + (n-1)$

$T(n) = (n-1) + (n-2) + \ldots = n(n-1)/2$

Similar problem if A is nearly sorted.

Is this unlikely?

Can a "hack" help?

E.g., splitter = median(first, middle, last)?

Average-case Quicksort complexity

- Intuitively, hope that on "random" input, most of the splits aren't too uneven.
  - If, say, 50% of time, splits are no worse than 1/10 vs 9/10, you might be OK
    - As long as the bad splits are evenly spread around, this kind of looks like the recurrence:

      $T(n) < T(n/10) + T(9n/10) + 2n$
    - Actually, on random input, things are more even.
    - But this is far from a proof!

Digression: Probability

A sample space S is a set of "elementary events".

An event is a subset of S.

A probability distribution is a function Pr from events to real numbers in [0,1] which satisfies certain properties.

If S is discrete (finite or countably infinite), these properties amount to:

- If $s \subseteq S$, $Pr\{s\} = \Sigma\, Pr\{e\}$, where sum is over $e \in s$.
- $Pr\{S\} = 1$.

If $|S| = n$ and $Pr\{e\} = 1/n$ for all $e \in S$, Pr is called uniform.

Note: We write Pr{s} or Pr{e} rather than Pr(s) or Pr({e}).

Probability factoids

Probabilities are often abused

What does "There's a 30% chance of rain" mean ??

It is not possible to have a uniform probability space on $\mathbb{N}$ or $\mathbb{Z}$ (or any countably infinite set).

- "Pick $n \in \mathbb{N}$ with equal probability" is meaningless.

There are 3 reasons for using uniform probabilities:

1. You control selection of events and ensure uniformity.
2. Someone else assures uniformity (you can shift blame.)
3. You can't think of anything better.

Reason #3 is a lousy reason!!

Random Variables

A (discrete) <u>random variable</u> X is a function from elementary events in a sample space to $\mathbb{R}$.

Examples:

- $X(p)$ = p's height for $p \in S$ = people in this class.
- $T_n(i)$ = time to solve instance i of size n problem.

Notation: "$X>72$" is the *event* $\{ p \in S \mid X(p)>72 \}$.

$X+Y$ is function $(X+Y)(e) = X(e) + Y(e)$.

The expectation $E[X]$ of X is $\sum_{e \in S} X(e) \Pr\{e\}$

$E[X]$ is the average, weights by the probabilities.

Theorem: $E[X+Y] = E[X] + E[Y]$.

Average-case Quicksort complexity

Important insight:

Let x & y be two elements with $x \le y$.

- If we pick some pivot z s.t. $x \le z \le y$ *before* we pick either x or y, then we'll *never* compare x to y.
- Assume we pick the pivots randomly from all the candidates in an unpartitioned group.
- If x and y are k places apart in the final sorted order, there chance of picking one of x or y before picking z between them is $2/(k+1)$.
- So the further apart two elements are in the final order, the less likely they will be compared.

Average-case Quicksort complexity

"If x and y are k places apart in the final sorted order, there chance of picking one of x or y before picking something between 2/k+1."

There are: 1 pair n-1 apart, $1 \times 2/n$

2 pairs n-2 apart, $2 \times 2/(n-1)$

...

n-1 pairs 1 apart, $(n-1) \times 2/2$

Expected number of comparisons is the total

(Uses fact: $E(\Sigma C_{xy}) = \Sigma E(C_{xy})$.)

C_{xy} is a "random variable"
= 1 if x and y are compared
0 otherwise

This total is $\Theta(n \lg n)$

Randomized algorithm

- Quicksort has "bad" problem instances.
- A randomized algorithm makes random choices *after* the instance is selected.
  - E.g. it can choose splitter via coin-flipping.
  - Algorithm can ensure each possible choice has equal probability.
  - An adversary can't find any particularly bad instance.

Probabilistic vs Average Analysis

Average-case analysis:

Let $P_n = \{I_1, I_2, ..., I_k\}$ be instances of size n

P_n is the sample space.

Uniform probabilities assumed.

Is there a good reason for this?

Average performance T(n) is $E(P_n)$.

Probabilistic analysis:

For each instance I, let $P_I = \{r_1, r_2, ..., r_k\}$ be possible choices made in randomized part of algorithm.

Assume uniform probabilities on P_I. (Is there a good reason?)

Define Time(I) = $E(P_I)$ (the expected time on instance I).

Probabilistic T(n) is max{Time(I)} max is over all instances of size n.

Probabilistic analysis of randomized Quicksort

- For each instance of sorting, randomized Quicksort has expected time $\Theta(n \lg n)$.
  - The same analysis as for that average time of (non-randomized) Quicksort.
- Warning: Result only holds for "truly" random choices of pivot elements.

Amazing paper by Karloff & Raghavan shows:

for any standard linear congruential pseudo-random number generator (e.g. Unix's "rand"),

there is a (carefully constructed) "bad" sorting instance

that, averaged over all PRNG "seeds"

has expected time $O(n^2)$

Stability

A sorting algorithm is <u>stable</u> if elements with equal keys stay in the same order.

If Sort({5, 8, 3, 10, 8}) returned {10, 8, 8, 5,3}, it wouldn't be stable (since 8 and 8 got swapped).

Stable sorting is useful; e.g. you might want to first sort by one key field, then by another.

- Is Heapsort stable?
- Is Quicksort stable?

Why use Quicksort?

- "Quicksort has tight code, so the hidden constant factor in its running time is small" (Text, pg 125).
  - Doesn't heapsort also??
- "[Quicksort] works well even in virtual memory environments." (Text, pg 145).
  - We'll see what that means!

Glossary (in case symbols are weird)

$\subseteq \epsilon \leq \forall \exists \Theta \Omega \Sigma \geq \cong * \neq \mathbb{N} \mathbb{R} \mathbb{Q} \mathbb{Z}$

$\subseteq$ subset ϵ element of

$\forall$ for all $\exists$ there exists

Θ big theta Ω big omega Σ summation

$\geq$ >= $\leq$ <= $\cong$ about equal

$\neq$ not equal $\mathbb{N}$ natural numbers(N)

$\mathbb{R}$ reals(R) $\mathbb{Q}$ rationals(Q) $\mathbb{Z}$ integers(Z)