Limits of FA

Can FA recognize all "computable" languages?

NO! Ex. $\{ 0^n1^n \mid n \geq 0\}$ is not regular

Intuitively, would have to keep track of no. of 0's so far, then check have same no. of 1's. These numbers are not limited, and so cannot build it into FA states.

Need new technique in order to prove!

Pumping Lemma

Intuitive idea: Consider a string accepted by a FA.

If the string is very long, then the states we go through must repeat, since the FA only has a fixed number of states.

Then we can take a snippet out of the string, and still accept.

Or, we can repeat the snippet any no. of times, and accept.

Pumping Lemma

If A is a regular language, then there is a no. p *(pumping length)* **where, if s is any string in A of length at least p, s may be divided into three pieces x,y,z, s = xyz, such that all of the following hold:**

1. **for each $i \geq 0$, xy^iz is in A**
2. **$|y| > 0$**
3. **$|xy| \leq p$**

Condition 1 lets us "pump out" elements in A

Note that x or z can be ε, but y cannot be (by 2)

(Without 2, the lemma is trivially true, with $y = \varepsilon$)

Condition 3 assures us we can make x and y small, if needed.

Alternating Quantifers in the Pumping Lemma

1. For each regular language L

(Choice of L)

2. There exists a pumping length p for L

(given p by lemma)

3. For every string s in L of length $\geq$ p

(Choice of s)

4. There exists x, y, z, with s = xyz, |y| > 0 and |xy| $\geq$ p.

(lengths y and xy restricted)

5. For each i $\geq$ 0 xy^iz in L.

(Choose an i that leads to contradiction)

Pumping Lemma Example

L = $\{0^n1^n \mid n \geq 0\}$ is not regular.

Suppose L were regular. Then let p be the pumping length given by the pumping lemma.

Let s = 0^p1^p in L. Note that |s| > p, so s = xyz with

1. for each i $\geq$ 0, xy^iz is in L
2. |y| > 0

(Don't use condition 3)

Case 1: y = 0 ... 0

Then xyyz will have more 0's than 1's,

so it cannot be in L, a contradiction.

Pumping Lemma Example

Case 2: y = 1 ... 1

Then xyyz will have more 1's than 0's,

so it cannot be in L, a contradiction.

Case 3: y = 0 ... 0 1 ... 1

Then xyyz will have 0's and 1's out of order, with some 1's before 0's, a contradiction.

Since these are all possible cases, we can conclude that L is not regular.

Pumping Lemma Example, using 3.

L = $\{ 0^n 1^n \mid n \geq 0\}$ is not regular.

Suppose L were regular. Then let p be the pumping length given by the pumping lemma.

Let s = $0^p 1^p$ in L. Note that |s| > p, so s = xyz with

1. for each i $\geq$ 0, $xy^i z$ is in L
2. |y| > 0
3. |xy| $\leq$ p

It must be the case that y = 0 ... 0, since xy is shorter than p.

But then xyyz will have more 0's than 1's,

so it cannot be in L, a contradiction.

Proving Languages Non-Regular

1. Assume L is regular. Then the Pumping Lemma holds.
2. Let p be the pumping length for L given by the lemma.
3. Choose a string s in L of length $\geq$ p.
   (Note that not every string in L will work!)
4. Consider all cases s can be divided into x, y, z, s = xyz , satisfying conditions 2 and 3.
   Show for each case that there is an i $\geq$ 0 with xy^iz not in L.
5. This provides a contradiction to the assumption of the pumping lemma, and to L being regular.

Examples: Showing Language L Non-Regular

L = {w | w has an equal no of 0 and 1's} *(1.39 p.80)*

1. Assume L is regular. Then the Pumping Lemma holds.
2. Let p be the pumping length for L given by the lemma.
3. We choose s= 0^p1^p (in L of length $\geq$ p).
4. Consider all cases s can be divided into x, y, z, s = xyz , satisfying conditions 2 (|y| > 0) and condition 3 (|xy| $\leq$ p).
   For this s, it must be that $y = 0^k$; this is the only case. We choose i=2: xy^2z will have more 0's than 1's, and so cannot be in L.
5. Contradiction.

Examples: Showing Language L Non-Regular

L = {ww | w in {0, 1} * } *(1.40 p.81)*

1. Assume L is regular. Then the Pumping Lemma holds.
2. Let p be the pumping length for L given by the lemma.
3. We choose s= 0^p10^p1 (in L of length $\geq$ p).
4. Consider all cases s can be divided into x, y, z, s = xyz , satisfying conditions 2 (|y| > 0) and condition 3 (|xy| $\leq$ p).
   For this s, it must be that y = 0^k; this is the only case. We choose i=2: xy^2z will have more 0's before the first 1 than the second 1; not in L.
5. Contradiction.

Examples: Showing Language L Non-Regular

L = {0^i1^j| i > j} *(1.42 p.82)*

1. Assume L is regular. Then the Pumping Lemma holds.
2. Let p be the pumping length for L given by the lemma.
3. We choose s= $0^{p+1}1^p$ (in L of length $\geq$ p).
4. Consider all cases s can be divided into x, y, z, s = xyz , satisfying conditions 2 (|y| > 0) and condition 3 (|xy| $\leq$ p).
   For this s, it must be that y = 0^k; this is the only case. We choose i=0: xz does not have more 0's than 1's, and so cannot be in L. *(i = 2?)*
5. Contradiction.

Another Example

Show $\{ a^n b^n c^n \mid n \geq 0\}$ is not regular.

Show $\{ a^n b a^m b a^{n+m} \mid n, m \geq 1\}$ is not regular.