Math 96:
Homework 7

Fall 2023

This homework is due in class on Friday, December 1st. Please complete at least one problem below.

1993 A3: Let $\mathcal{P}_n$ be the set of subsets of $\{1, 2, \ldots, n\}$. Let $c(n, m)$ be the number of functions $f : \mathcal{P}_n \to \{1, 2, \ldots, m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n, m) = \sum_{j=1}^{m} j^n.$$

1993 A4: Let $x_1, x_2, \ldots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y_1, y_2, \ldots, y_{93}$ be positive integers each of which is less than or equal to 19. Prove that there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's.

1993 B2: Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$. The last person to discard wins the game. Assuming optimal strategy by both A and B, what is the probability that A wins?

2003 A1: Let n be a fixed positive integer. How many ways are there to write n as a sum of positive integers,

$$n = a_1 + a_2 + \cdots + a + k,$$

with k an arbitrary positive integer and $a_1 \leq a_2 \leq \cdots \leq a_k \leq a_1 + 1$? For example, with $n = 4$ there are four ways: $4, 2 + 2, 1 + 1 + 2, 1 + 1 + 1 + 1$.

1953 A2: Six points are in general position in space (no three in a line, no four in a plane). The fifteen line segments joining them in pairs are drawn and then painted, some segments red, some blue. Prove that some triangle has all its sides the same color.

1958 November B1: Given

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^{-1}, n \geq 1,$$

prove that

$$b_n = \frac{n+1}{2n} b_{n-1} + 1, n \geq 2,$$

and hence, as a corollary,

$$\lim_{n\to\infty} b_n = 2.$$