Math 96:
Basic Problem Solving Techniques

September 29th, 2023

So you have a new problem to solve. Especially early on, it can be confusing as to what exactly to do with it. Here are some basic techniques to get started.

1 Read the Problem

No, seriously. Read the problem carefully.

- Do you know the meaning of all the words? If you do not, you might just need to skip this problem (a few Putnam problems will involve more advanced math).
- Can you restate the problem in your own words?
- Can you list all of the assumptions that you are allowed to make? Note: very often you will need to make use of every assumption in your proof.

2 Things to Try

Once you understand the problem, you then need to try to find a solution. Mostly, this involves stumbling around and trying new things until you either gain some insight or find something that works. Perhaps the single best piece of advice I can give you is this:

Don't give up. As long as you are trying new things, even if they don't work, you are making progress.

But sometimes you get stuck and don't know what to do. If you don't know what to try next, here are some general techniques to fall back on. Not all of these will apply to all problems, but for many problems at least one of these will give you a good start in whatever you are working on.

- Draw a picture.
- Try working small examples or special cases.
- Look for patterns.

- Guess and check (use induction).
- Give variable names to relevant quantities.
- See what you can compute or prove.
- Write down formulas. See what you can derive from them.
- Make conjectures. What seems to be true about this problem. Can you prove it?
- Work backwards. What would imply the problem statement? What else needs to be true if the problem statement is?
- Use proof by contradiction. Try to construct a counterexample- it won't work, but *why* it doesn't work is usually instructive.
- Reread the problem statement. Are there conditions that you aren't using?
- Use symmetry.
- Consider extremal cases. What can you say about the furthest away/largest/most whatever object in the problem?

3 How to Practice

There are two main ways to train yourself to be better at this kind of open ended problems. The first is to learn specific tricks and ideas, generally specific to one particular kind of problem. This is what our class's techniques lectures will be about. But perhaps the more important method is to simply practice working on problems.

When working on practice problems, while it is often useful to be able to ask for help or to look up answers, I would advise against resorting to these resources too quickly. I would recommend working on a problem for at least an hour or two on your own before seeking additional help. On the one hand, this gives you a chance to solve the problem on your own, and additionally having worked on the problem before looking up a solution will mean that you will have a better understanding of the solution and of what the important ideas in it are.

4 Examples

1958 November B6: Let a complete oriented graph on n points be given, i.e., a set of n points $1, 2, 3, \ldots, n$, and between any two points i and j a direction, $i \to j$. Show that there exists a permutation of the points $[a_1, a_2, a_3, \ldots, a_n]$, such that $a_1 \to a_2 \to a_3 \to \cdots \to a_n$.

1948 B3: If n is a positive integer, prove that

$$[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+2}],$$

where $[x]$ denotes as usual the greatest integer not exceeding x.

2013 A4: A finite collection of digits 0 and 1 is written around a circle. An *arc* of length $L \geq 0$ consists of L consecutive digits around the circle. For each arc w, let $Z(w)$ and $N(w)$ denote the number of 0's and the number of 1's in w, respectively. Assume that $|Z(w) - Z(w')| \leq 1$ for any two arcs w, w' of the same length. Suppose that some arcs $w_1, \ldots, w_k$ have the property that

$$Z = \frac{1}{k}\sum_{i=1}^{k} Z(w_i) \textbf{ and } N = \frac{1}{k}\sum_{i=1}^{k} N(w_i)$$

are both integers. Prove that there exists an arc w with $Z(w) = Z$ and $N(w) = N$.